Outer Code Designs for Augmented and Local-Global Polar Code Architectures [1]

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***Abstract*—In this paper, we introduce two novel methods to design outer polar codes for two previously proposed concatenated polar code architectures: augmented polar codes and local-global polar codes. These methods include a stopping set (SS) construction and a nonstationary density evolution (NDE) construction. Simulation results demonstrate the advantage of these methods over previously proposed constructions based on density evolution (DE) and LLR evolution.**

I. Construction Methods

In this section we describe the stopping set and NDE methods for designing outer codes for augmented polar codes. The stopping set construction is inspired by the stopping set analysis for conventional polar codes in [2]. Both methods can be extended to the local-global polar code architecture [3], but some care is needed. More details can be found in the full-length paper.

A. Stopping Set Construction

Consider the augmented polar code structure in Fig. 1, where an inner polar code is concatenated with an outer polar code. Let $v_0(i)$ denote the i-th node on the leftmost stage within the original outer factor graph, and $v_1(j)$ be the j-th node on the leftmost stage within the original inner factor graph. Let $\mathcal{J}_i$ denote the set of inner information nodes such that each element in $\mathcal{J}_i$ is connected to one of the leaves in the stopping tree $ST(i)$ defined on the original outer factor graph. Let $MVSS(\mathcal{J}_i)$ be the MVSS defined by set $\mathcal{J}_i$ on the original inner factor graph. Let $G_{\mathcal{J}_i}$ denote the submatrix of the inner encoding matrix $G = F^{\otimes n_1}$ consisting of the rows that correspond to $\mathcal{J}_i$. The following theorem gives a bound on the size of $MVSS(\mathcal{J}_i)$.

Theorem I.1. *Given any outer code position i on the leftmost stage of the factor graph and its corresponding $\mathcal{J}_i$, $1 \le i \le N_0$, we have $|MVSS(\mathcal{J}_i)| \ge g(G_{\mathcal{J}_i})$, where*

$$g(A_{m\times n}) = \sum_{j=1}^{n} \delta(\sum_{i=1}^{m} a_{ij} - 1) \tag{1}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

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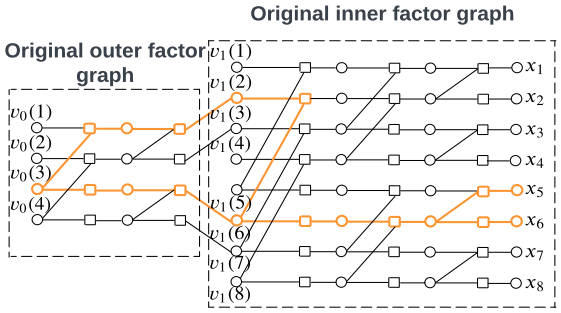

Fig. 1. Augmented structure with N_0=4, N_1=8.

Theorem I.1 suggests a practical way to design outer polar codes based on the size of stopping sets. We first initialize an unfrozen set $\mathcal{O}$ for the outer code using the conventional DE algorithm, for example. Then we swap a specified number of unfrozen bits $i \in \mathcal{O}$ with the smallest "stopping distance" $g(G_{\mathcal{J}_i})$ with some positions $j \in \mathcal{O}^c$ such that $g(G_{\mathcal{J}_j}) > g(G_{\mathcal{J}_i})$.

Let Q be a length N_0 vector that contains the indices of bit channels ordered according to channel reliability calculated by DE. The indices are ordered by descending channel reliability. Let s denote the number of bits we are going to swap. For convenience, we denote $g(G_{\mathcal{J}_i})$ by $g(i)$. Let K_0 denote the size of the desired unfrozen set. Let $min_s(\cdot)$ be the function that returns the s-th smallest value in a vector, while $min(\cdot)$ returns the smallest value along with its index. Note that s should be chosen such that there are more than s frozen bits that have $g(\cdot)$ value larger than $min_s(g(Q(1)), ..., g(Q(K_0)))$. The detailed swapping algorithm is presented in Algorithm 1.

B. Nonstationary Density Evolution (NDE)

In prior formulations of augmented polar codes in [4], the outer polar codes were designed independently from the code concatenation. Since the inputs to the outer code are the LLRs passed from the inner code and the LLR densities may vary within the semipolarized bit-channels, there is a potential mismatch with the assumed reliability ordering of the outer code bit-channels which could lead to inferior performance of the augmented code.

Algorithm 1 Stopping set construction

```
    Input: Q; g(i) for each i ≤ N_0; s
    Output: designed unfrozen set O
 1: threshold = min_s(g(Q(1)), ..., g(Q(K_0)))
 2: i ← 1
 3: while i ≤ s do
 4:     [value, index] = min(g(Q(1)), ..., g(Q(K_0)))
 5:     j ← 1
 6:     while True do
 7:         if g(Q(K_0 + j)) > threshold then
 8:             Q(index) ← Q(K_0 + j)
 9:             delete Q(K_0 + j) from Q
10:             jump to line 14
11:         end if
12:         j ← j + 1
13:     end while
14:     i ← i + 1
15: end while
16: Return O = Q(1 : K_0)
```

In order to improve the performance of the concatenated code under BP decoding, we introduce a small modification of the conventional DE design method which we call the nonstationary DE (NDE) algorithm. The NDE algorithm does not assume that the inital LLR distributions are identical; rather, the LLR distributions coming from the inner code at the rightmost stage of the outer code factor graph correspond to N_0 separate binary symmetric memoryless channels $W_i, i = 1, \ldots, N_0$.

The densities on the rightmost stage are initialized as $\mathrm{a}_1^i = \mathrm{a}_{W_i},\ \ 1 \le i \le N_0$. We then apply DE over the SC decoding trees to compute the LLR densities $\mathrm{a}_{N_0}^i, i = 1, \ldots, N_0$ for the bit-channels. Assuming symmetry and an all-zero codeword, we compute a probability of error, denoted $P(\mathcal{A}_i)$, for bit-channel i by integrating $\mathrm{a}_{N_0}^i$ over the interval $(-\infty, 0)$. The unfrozen set $\mathcal{O}$ is chosen to minimize $\sum_{i\in\mathcal{O}} P(\mathcal{A}_i)$, subject to $|\mathcal{O}| = K_0$.

In practice, we replace each initial LLR density of the outer code with the empirical LLR density of the corresponding bit-channel of the inner code after t iterations of BP decoding under assumption of an all-zero codeword. If the semipolarized bit channels of the inner code are denoted by index set $\mathcal{H}$, then the initial LLR density a_1^i is replaced by the empirical density $b_t^{\mathcal{H}(i)}$ of the corresponding semipolarized bit-channel $\mathcal{H}(i)$. Details of the NDE implementation are given in Algorithm 2.

II. Empirical results

This section gives empirical results under BP decoding for augmented and local-global polar codes constructed using the proposed methods, along with comparisons to codes designed using conventional DE and LLR evolution methods. Fig. 2 shows frame error rate (FER) results for augmented code constructions with component code lengths $N_0 = 64, N_1 = 1024$. Figs. 3 presents the results for global decoding for a local-global code with component code lengths $N_0 = 256, N_1 = N_2 = 1024$. In summery, both proposed methods improve error rate performance over conventional designs. More results can be found in the complete paper.

Algorithm 2 Nonstationary Density Evolution (NDE)

Input: $\mathcal{H}$; K_0; $b_t^{\mathcal{H}}$
Output: designed unfrozen set $\mathcal{O}$
$\mathrm{a}_1^i \leftarrow b_t^{\mathcal{H}(i)}$, for each $i \le N_0$
for column l from right to left **do**
$\triangleright$ on original outer factor graph
  for each node in column l **do**
    update its density by DE
  end for
end for
Return $\mathcal{O}$ which minimizes $\sum_{i\in\mathcal{O}} P(\mathcal{A}_i)$, $|\mathcal{O}| = K_0$

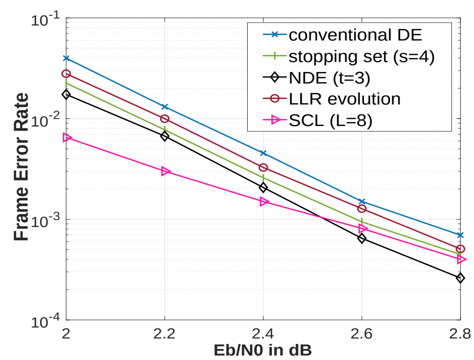

Fig. 2. Augmented polar code designs with N_0=64, N_1=1024.

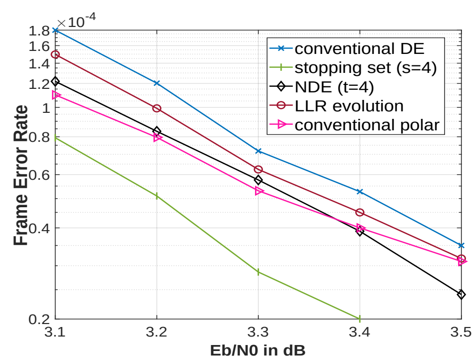

Fig. 3. Global decoding results for N_0=256, N_1=N_2=1024.

References

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