On the Capacity of DNA-based Data Storage under Substitution Errors

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May 9th, 2022

Outline

- Channel Model
- Related Work
- Preliminaries
- Channel Capacity
- Summary & Outlook

Data Storage in DNA

High density data storage

- DNA: $10^9\,\mathrm{GB/mm^3}$
- Tape: $10 - 100\,\mathrm{GB/mm^3}$

Data Storage in DNA

Never obsolete

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Data Storage in DNA

Long term data storage
(DNA from mammoths)

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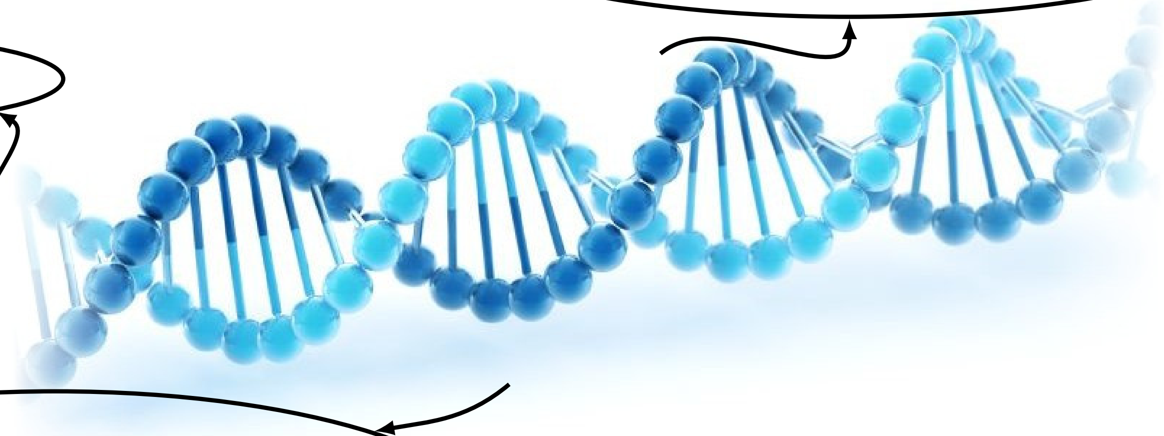

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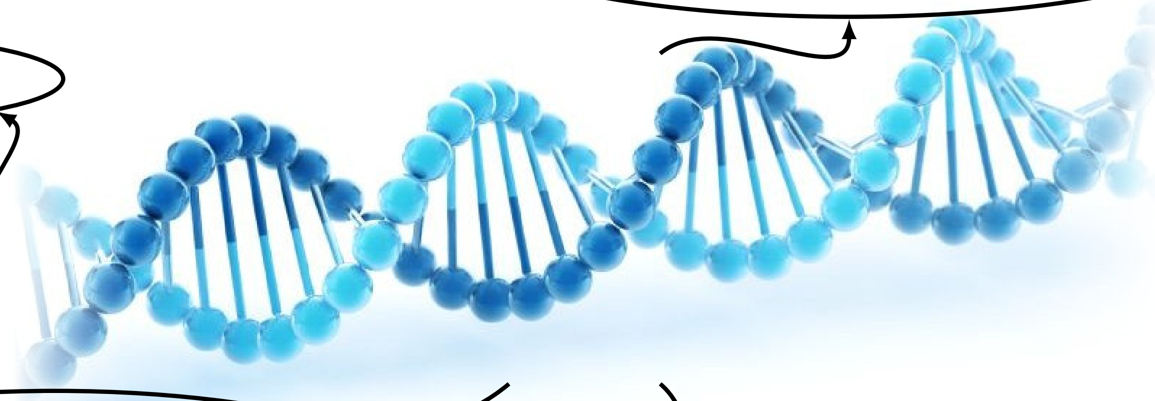

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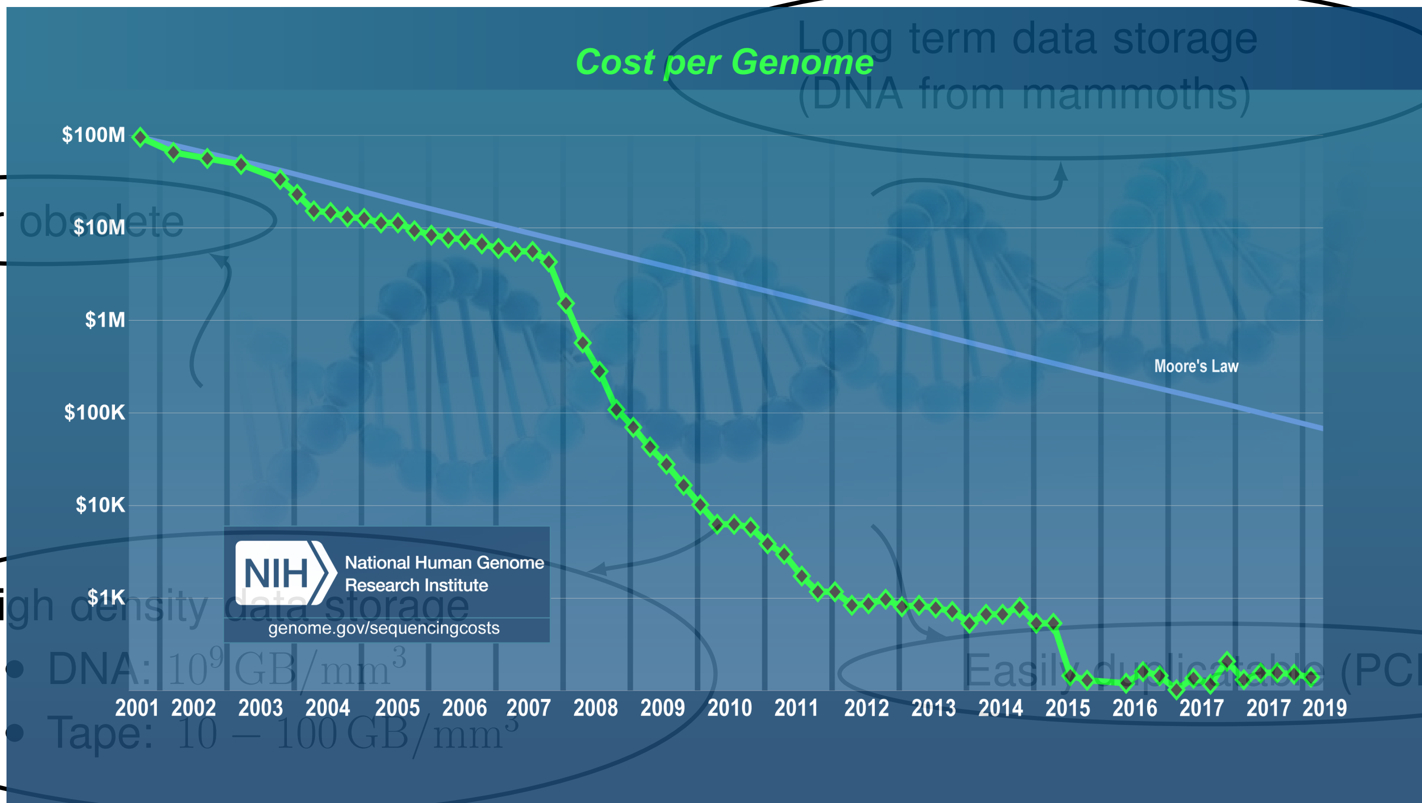
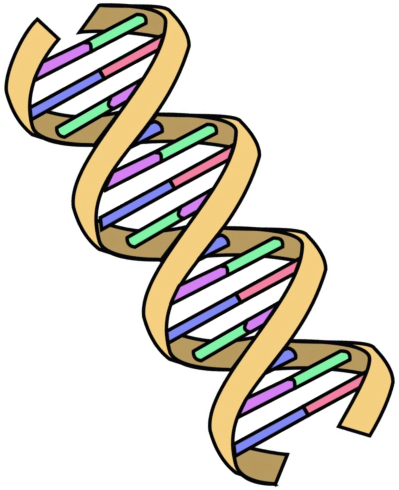


Data Storage in DNA

User Binary Data
000001101011001
110100010010101
101000111110100



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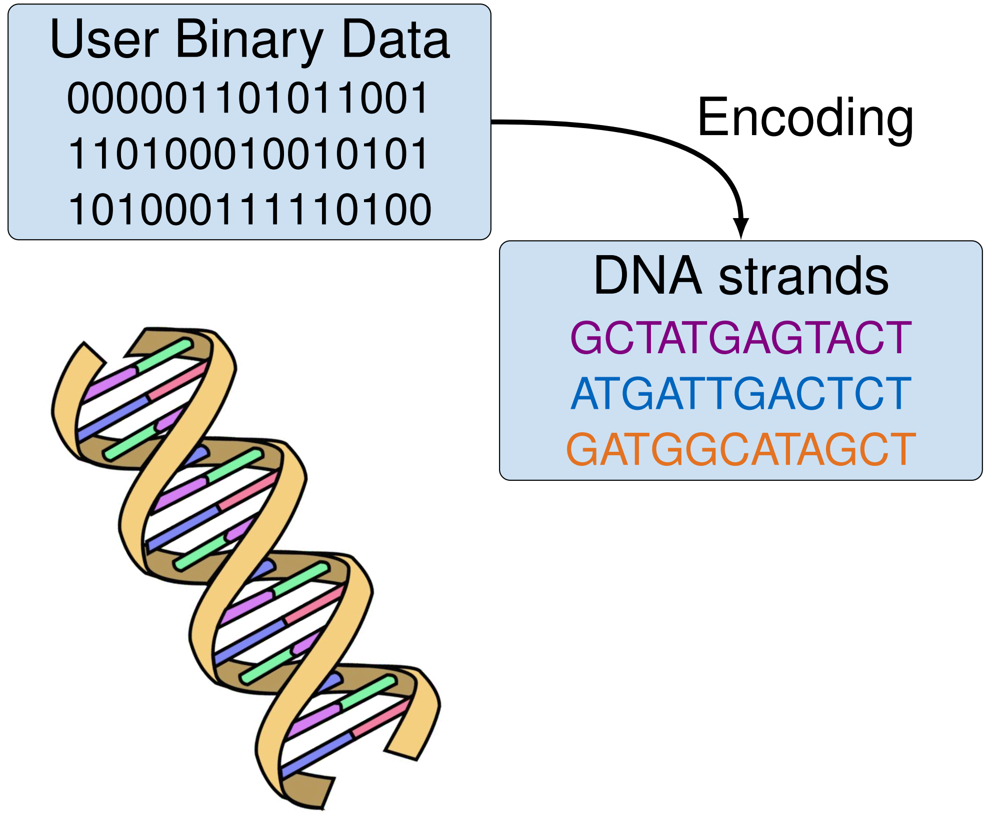

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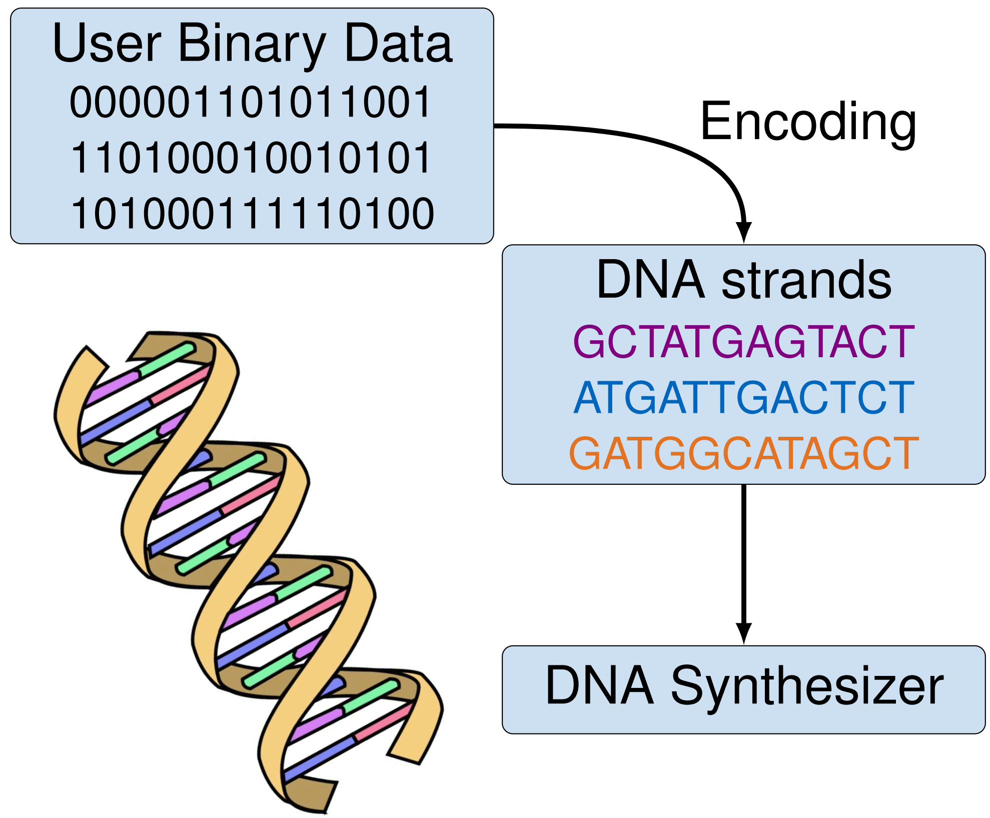

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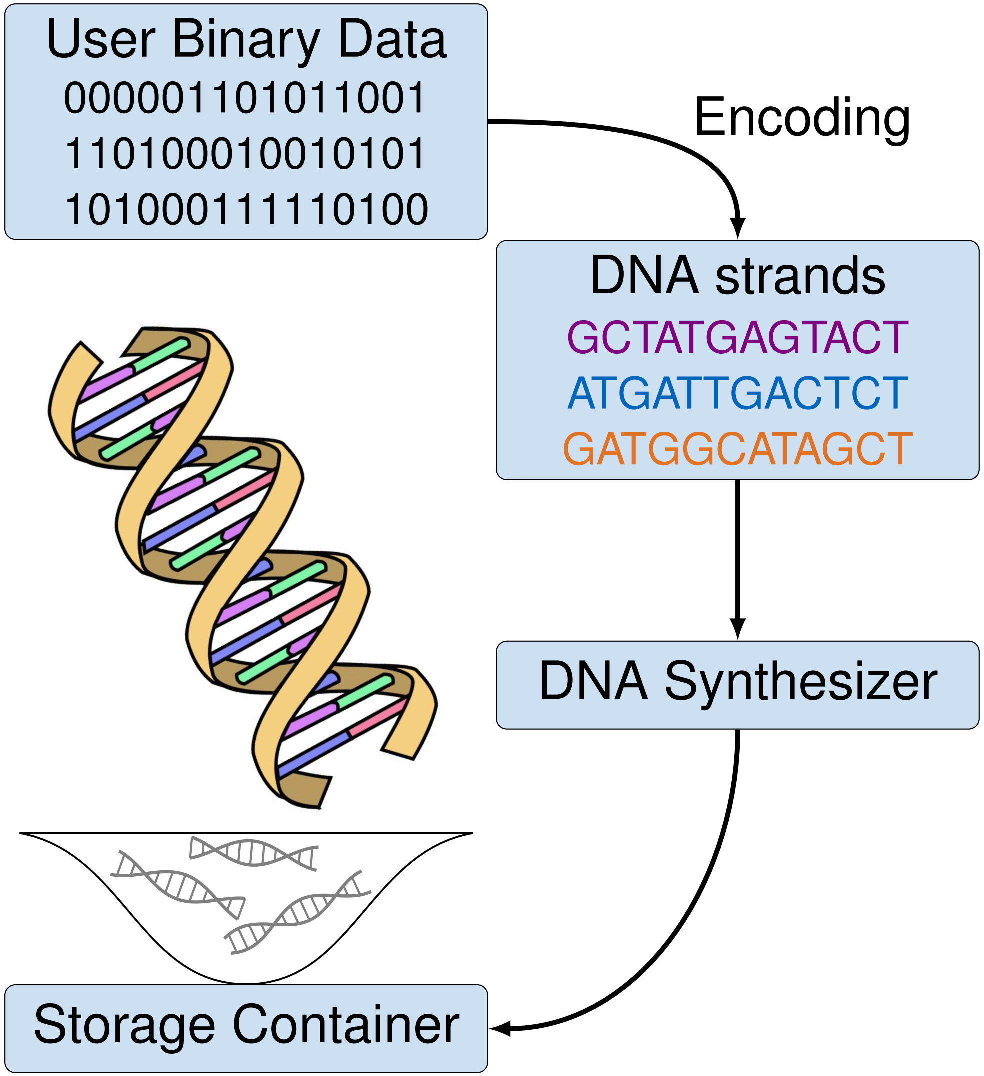

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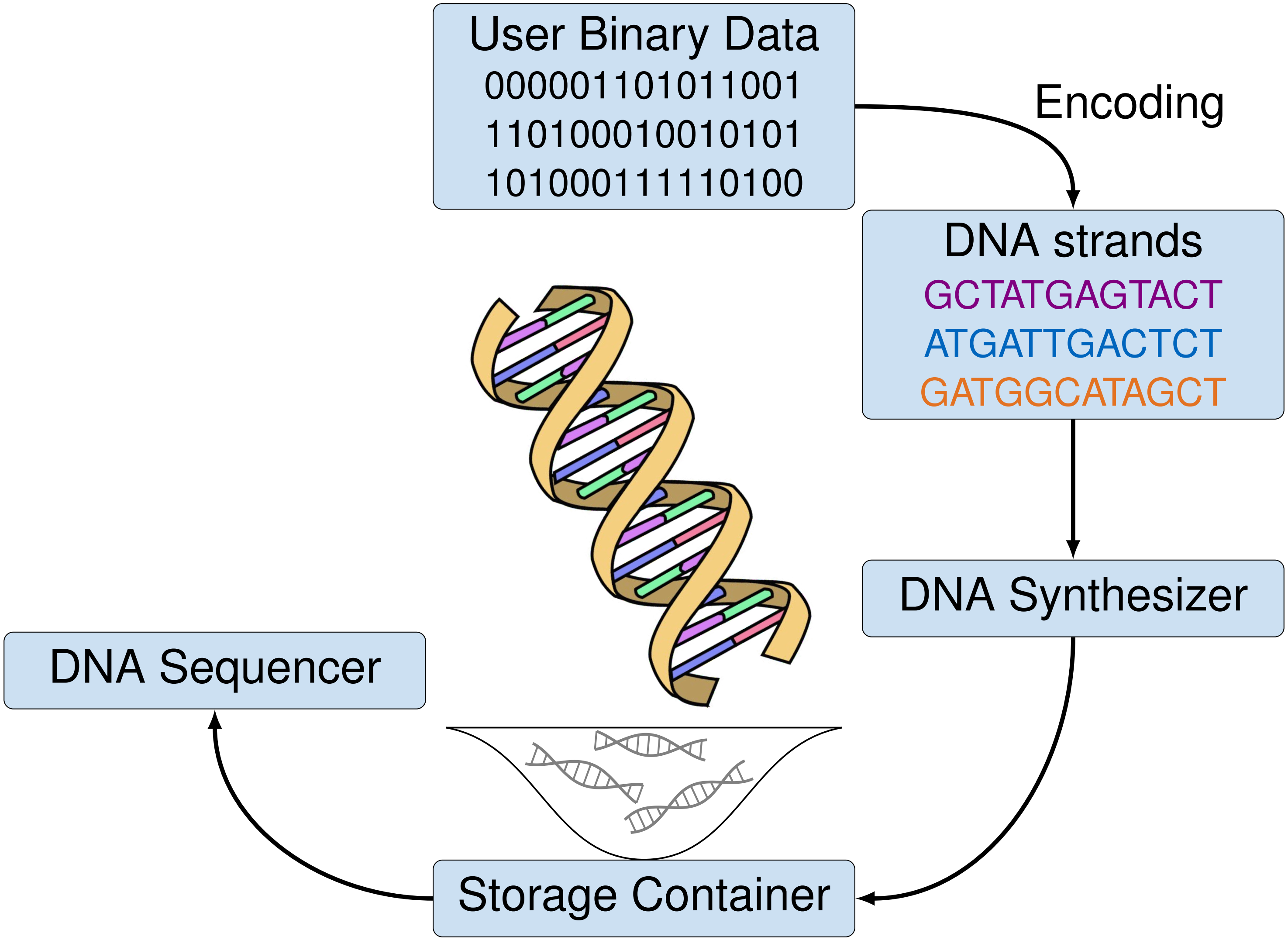

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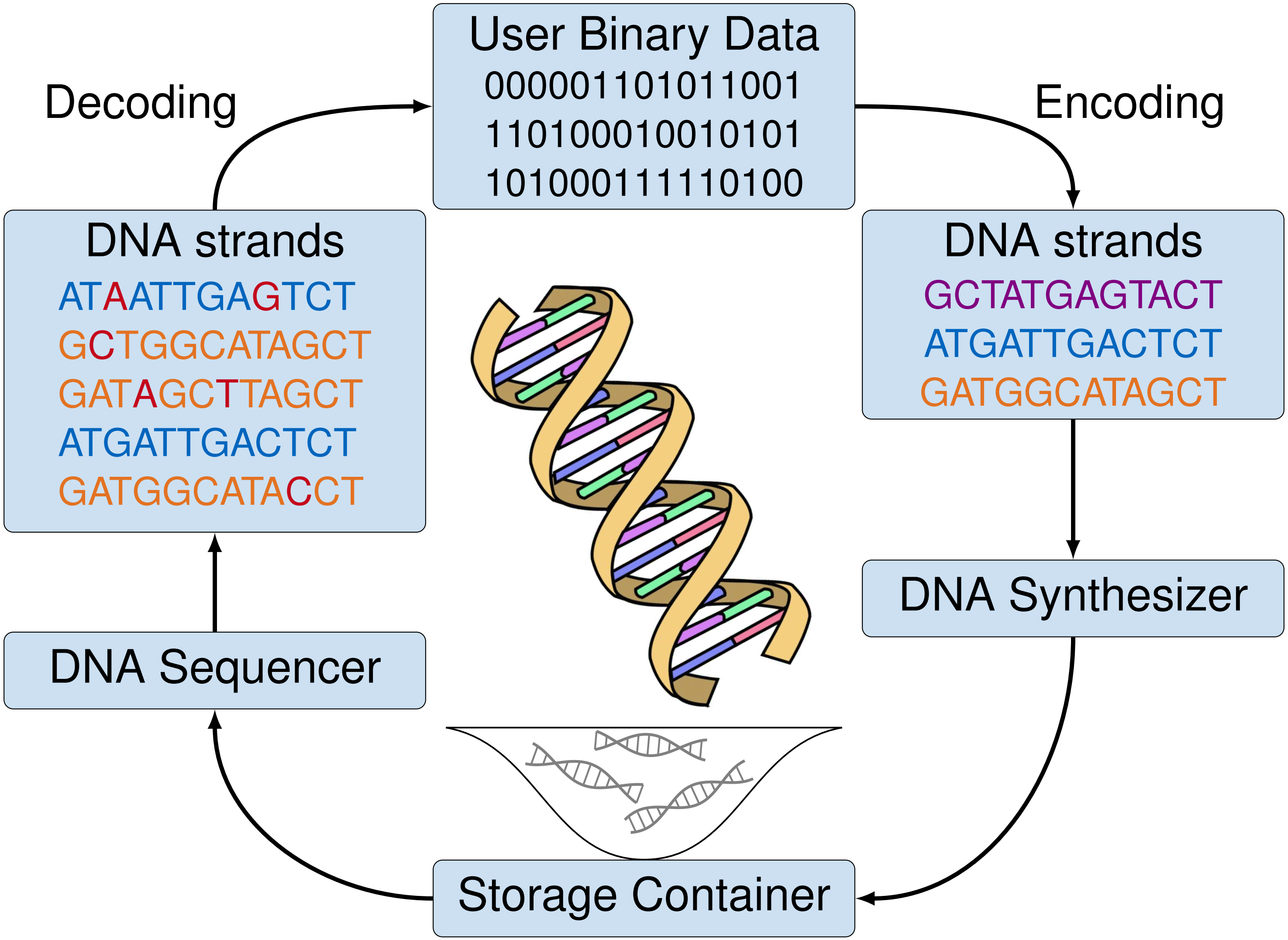


Channel Model

Y

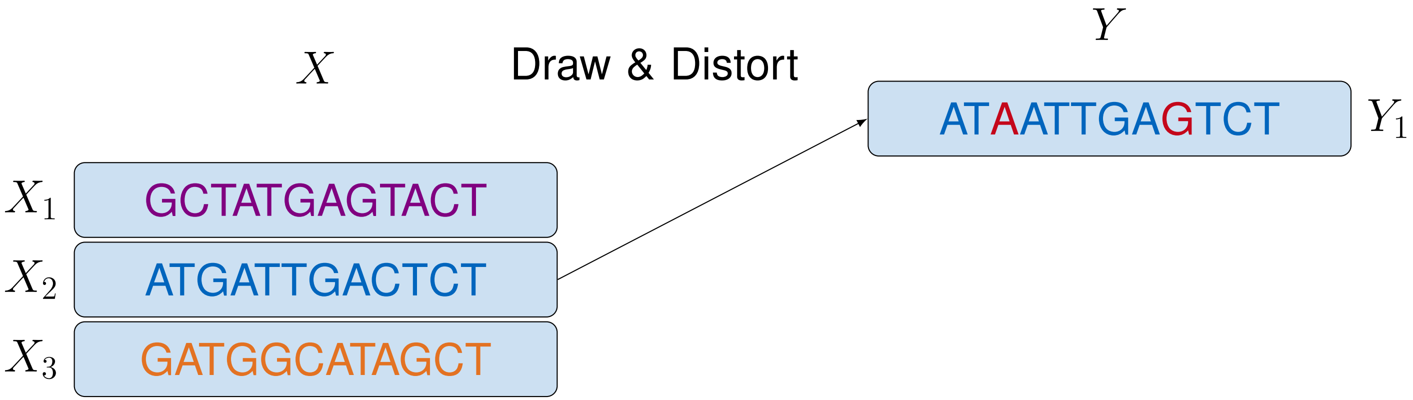
X Draw & Distort

X_1 GCTATGAGTACT

X_2 ATGATTGACTCT

X_3 GATGGCATAGCT

Channel Model



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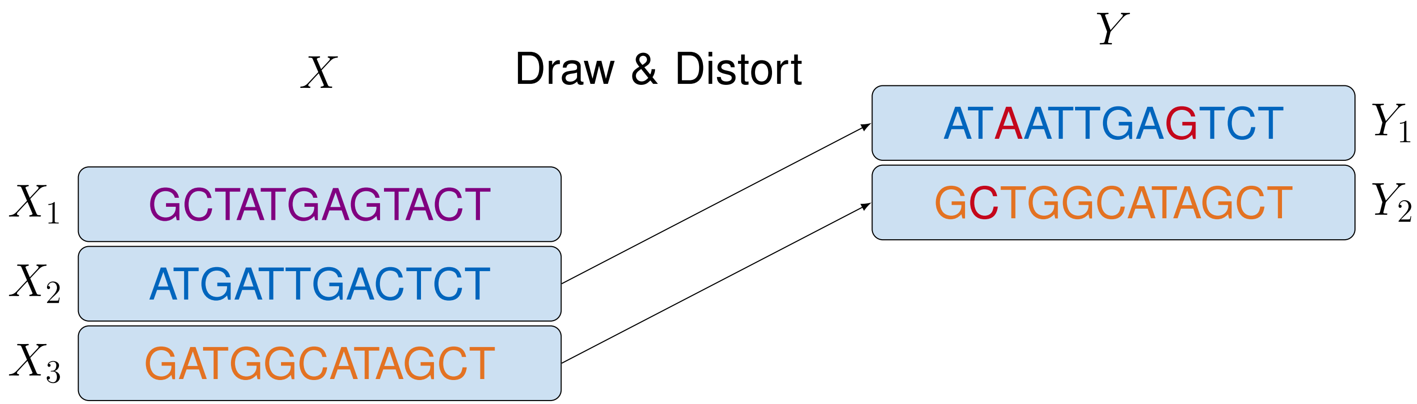

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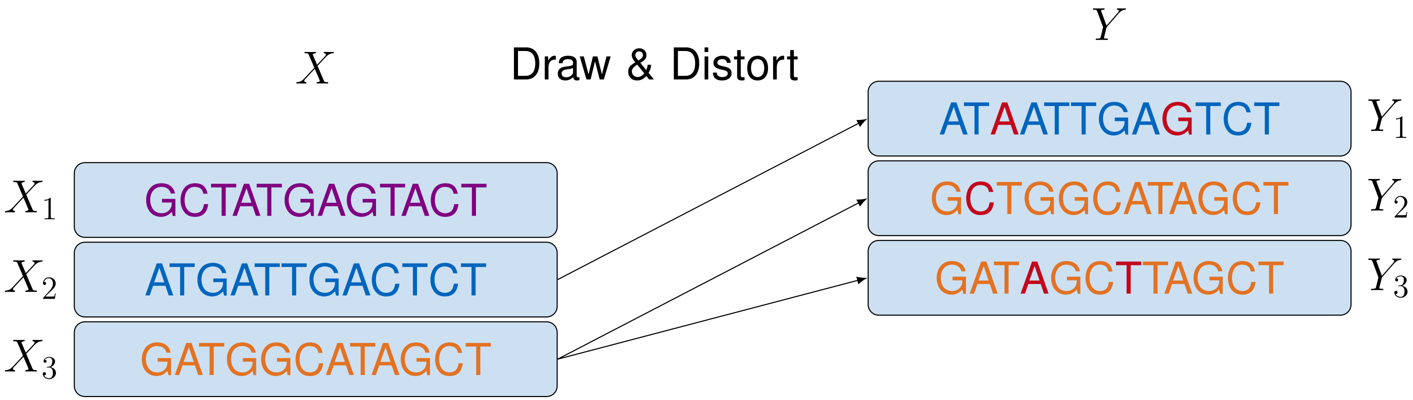

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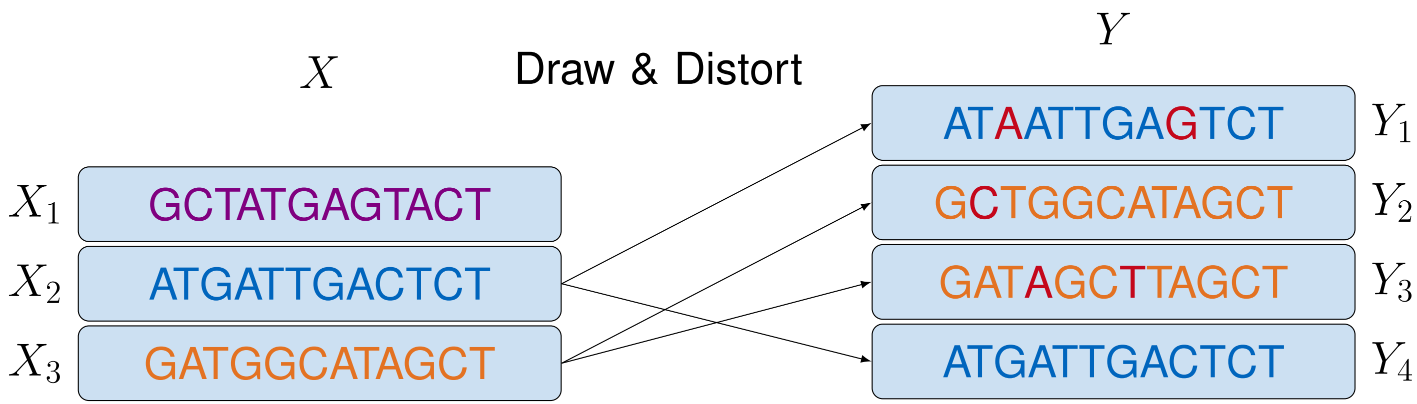

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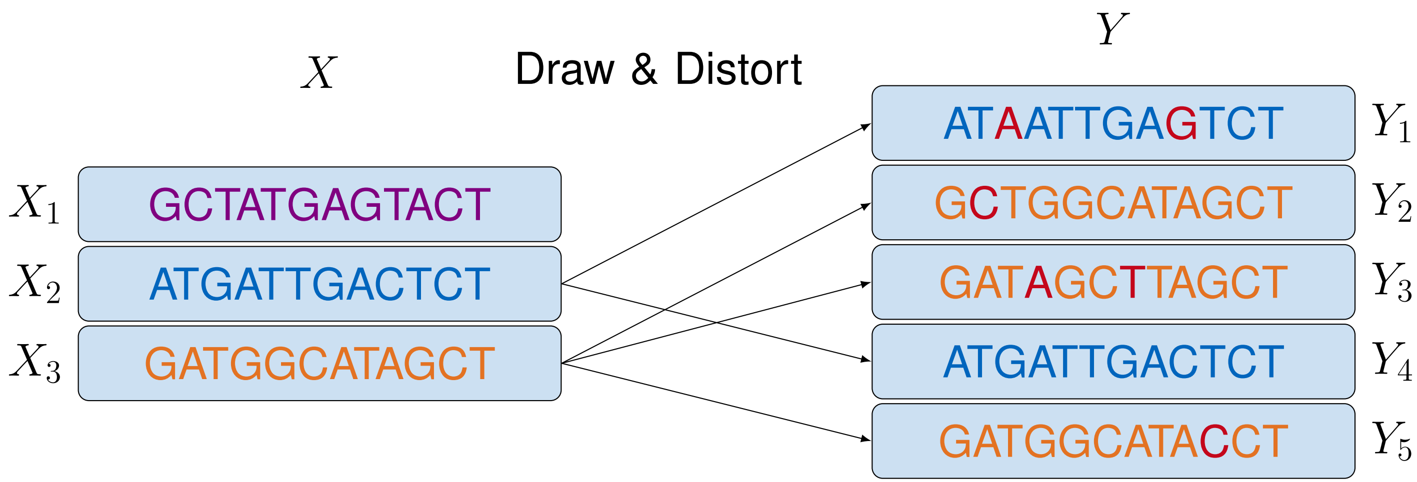

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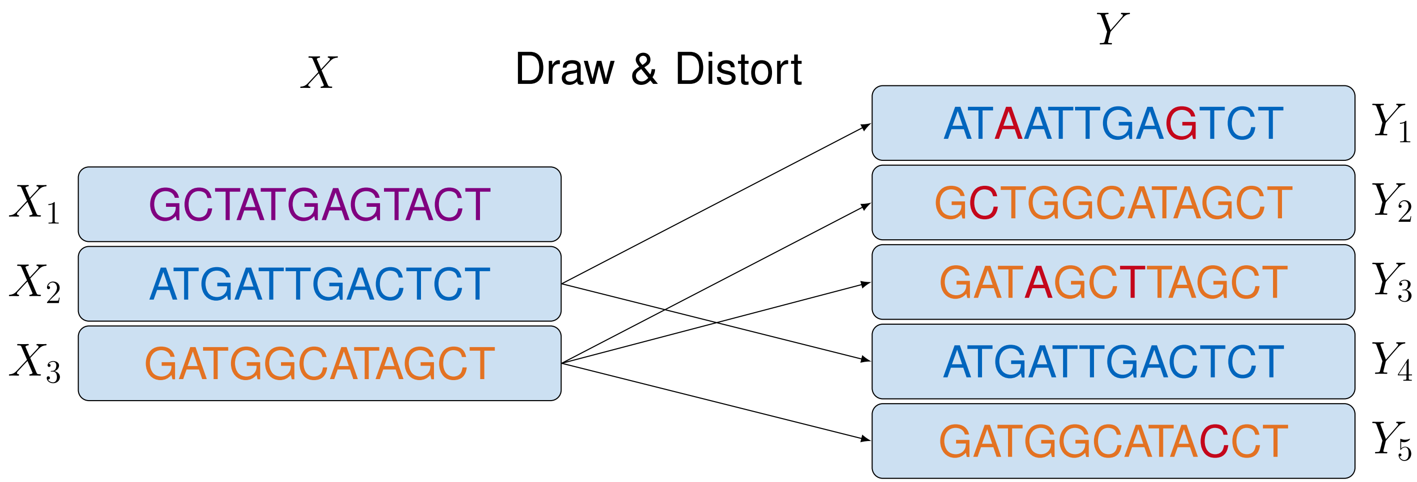

$$Y_j = X_{I_j} + E_j$$

- I_j : i.i.d. uniform random draws
- E_j : random error vectors (error probability p)

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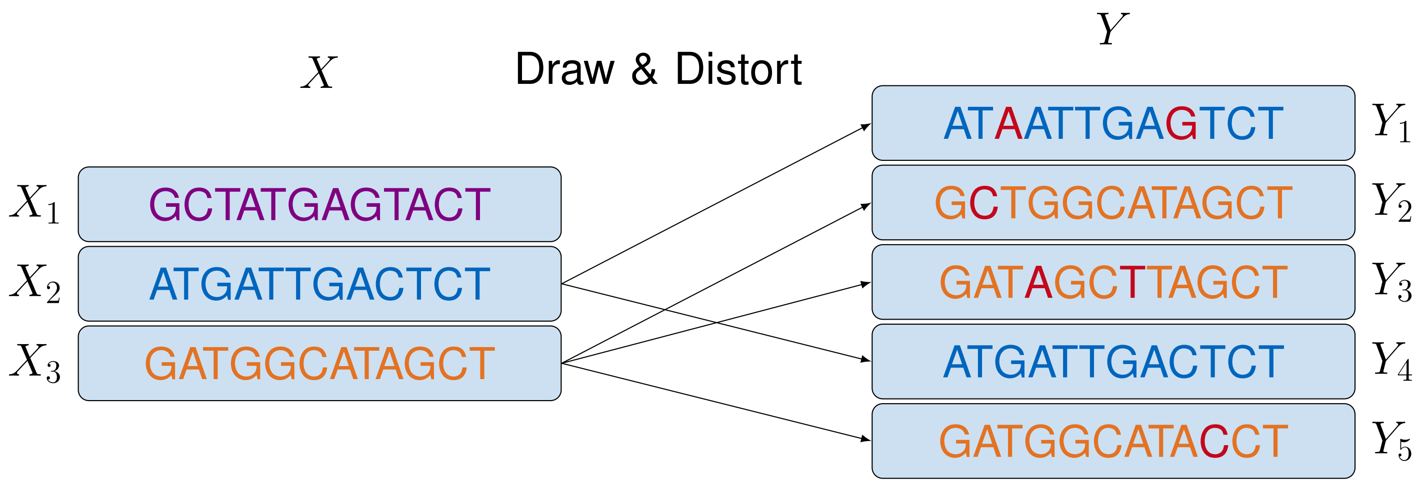

$$Y_j = X_{I_j} + E_j$$

- I_j : i.i.d. uniform random draws
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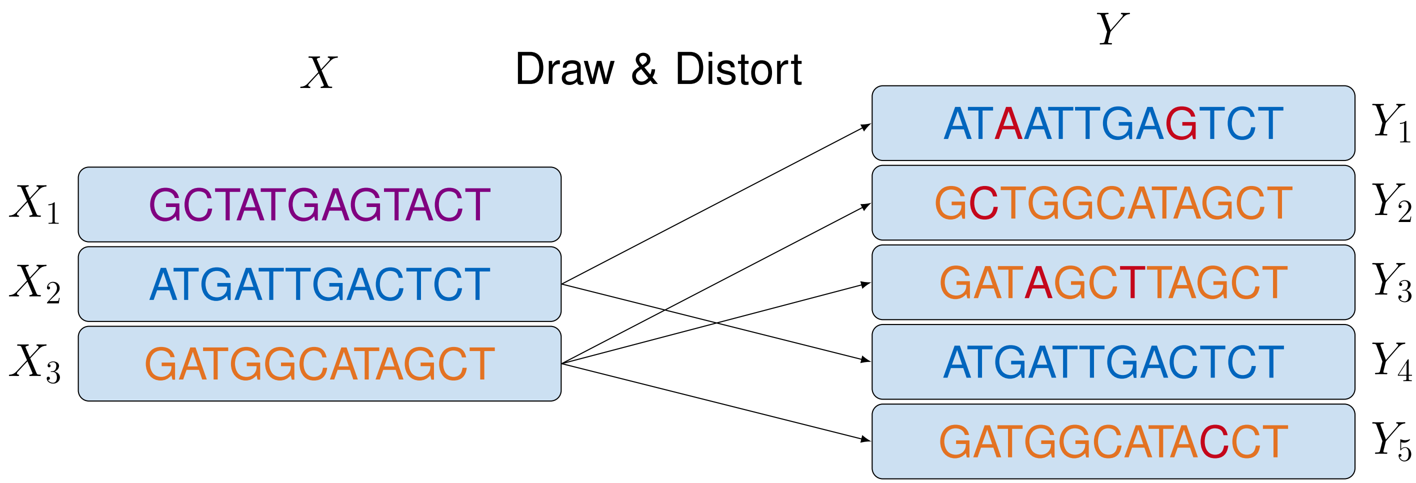
- In this work: Quaternary sequences ($\mathbb{Z}_4 = \{\mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{T}\}$)

Channel Model



Channel Input:

- M Sequences, each of length L
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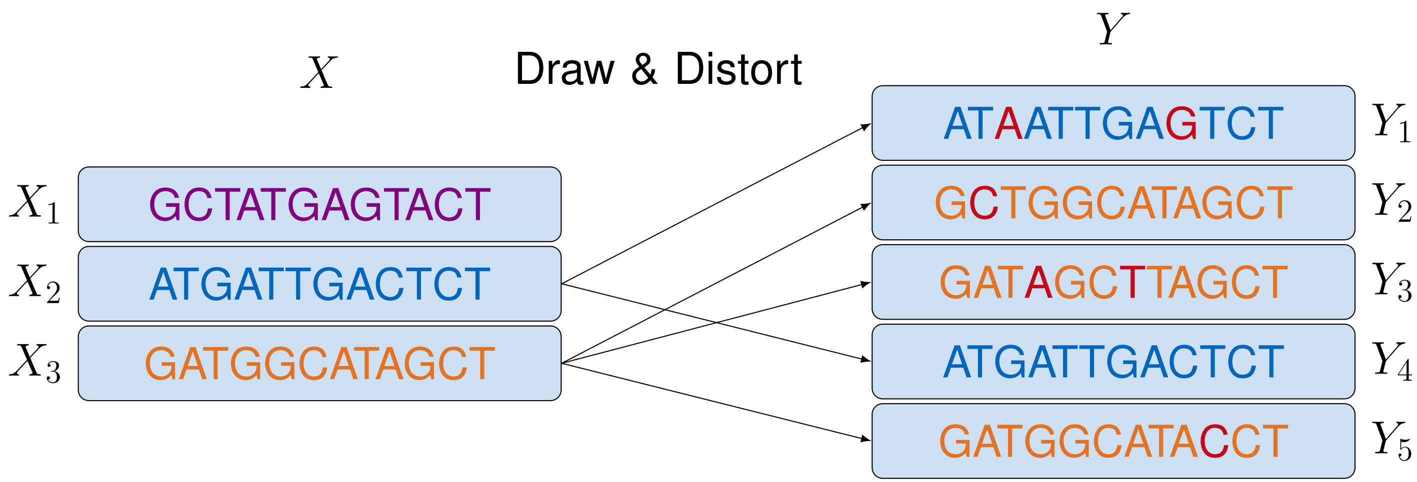
Channel Model



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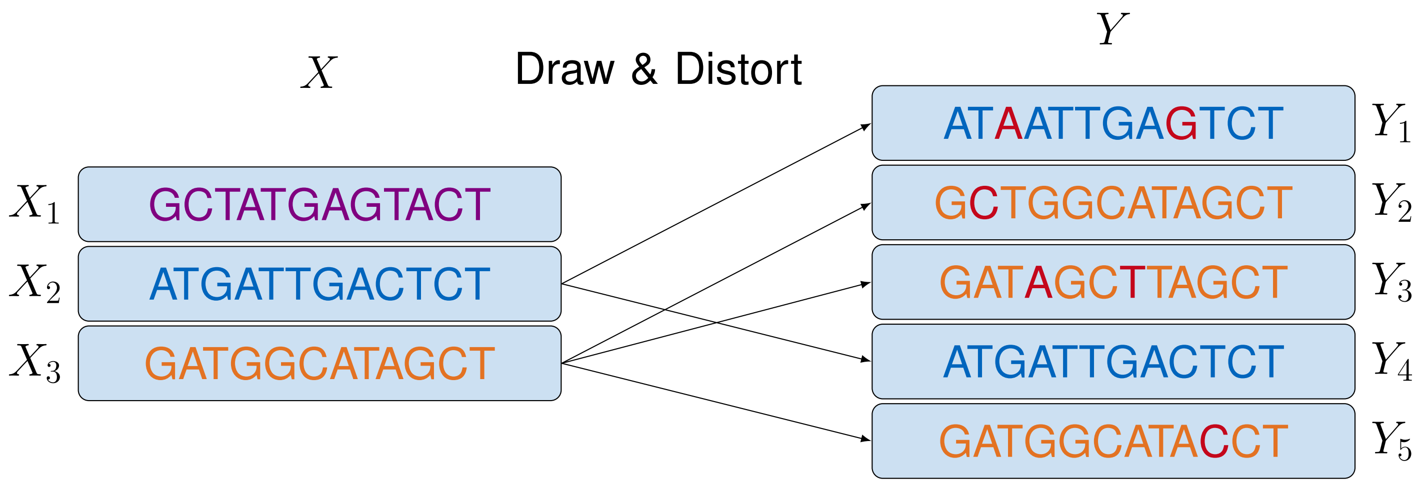

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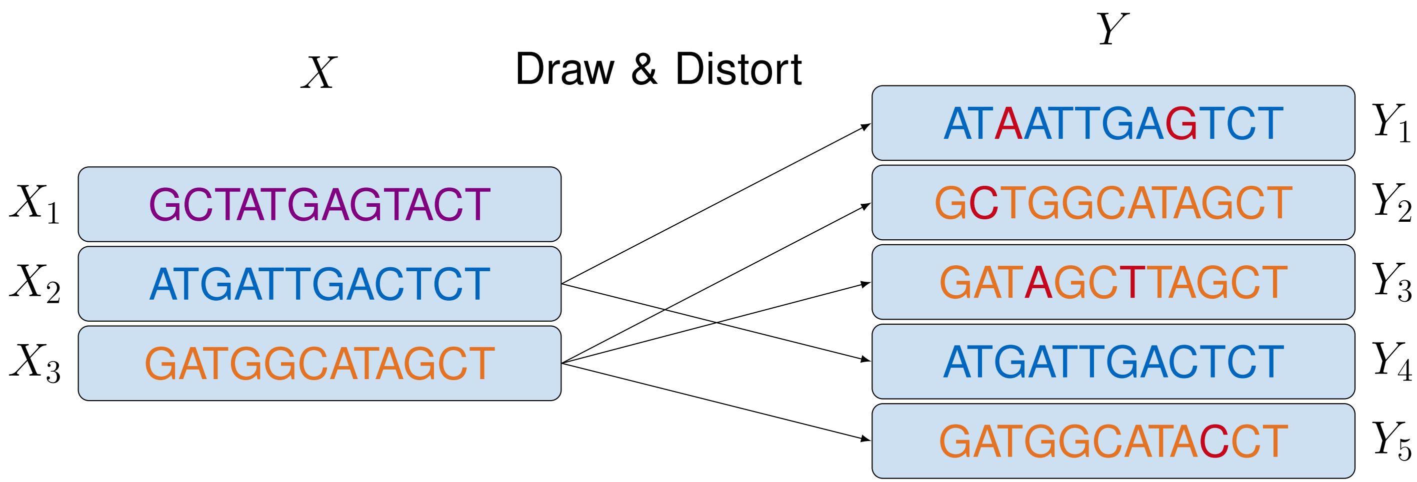

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- $c = N / M$

Channel Model - Codes and Information Rates

Communication System:

Message

W

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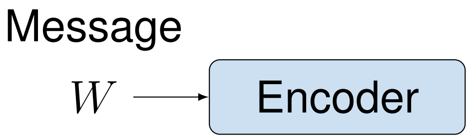

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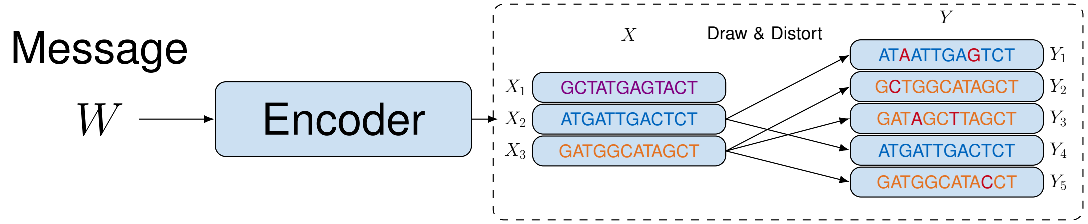

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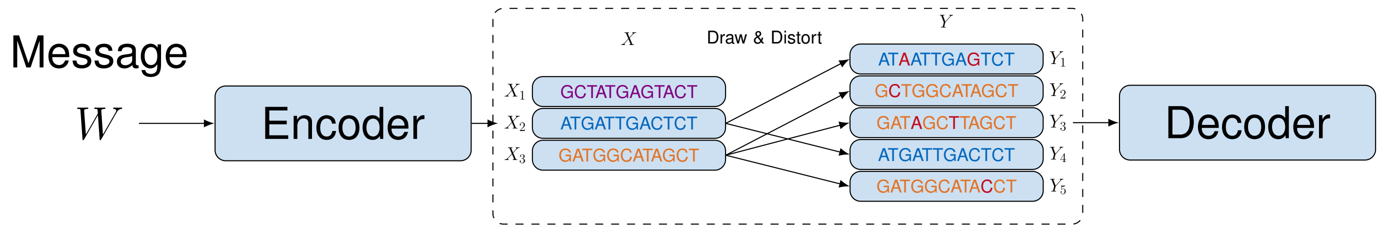

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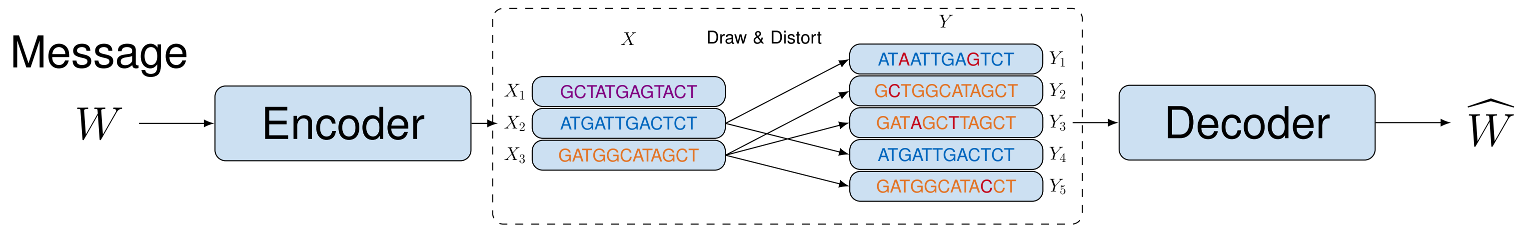

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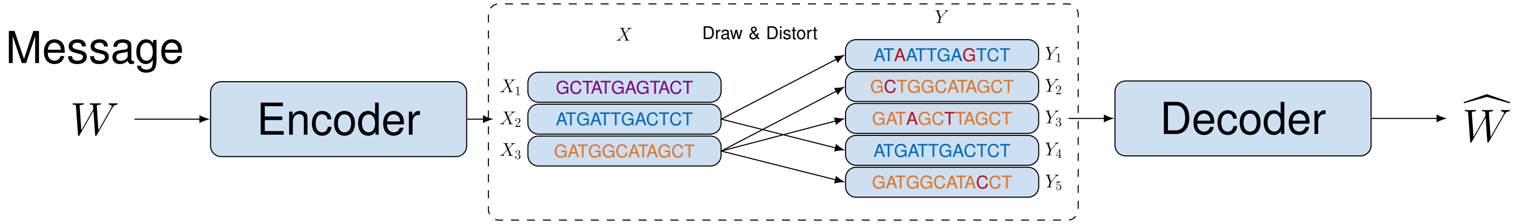

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- $\mathcal{C} = \{X(1), \ldots, X(4^{MLR})\} \subset \mathbb{Z}_4^{M \times L}$
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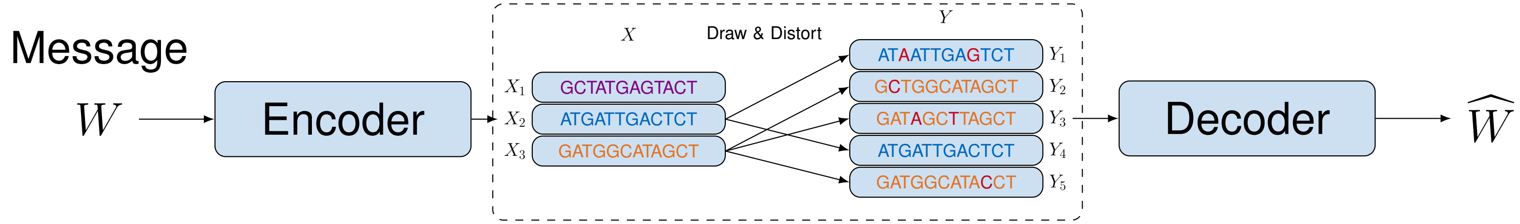

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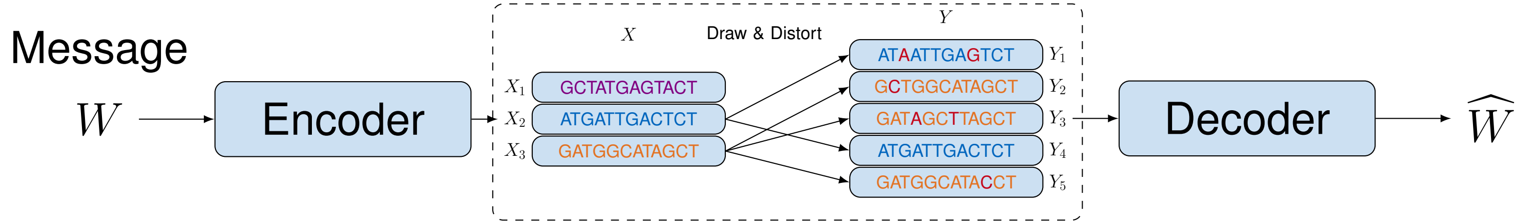

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Code rate R is achievable, if there exists a code $\mathcal{C}$ of rate R with $P(\text{Err}) \to 0$, as $ML \to \infty$

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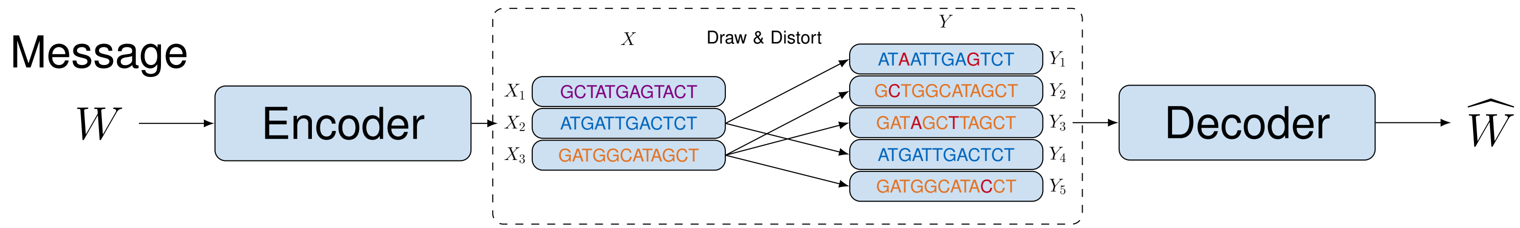

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- **Capacity**: Supremum of achievable rates

Related Work

[**Mitzenmacher,** ”***On the Theory and Practice of Data Recovery with Multiple Versions,” 2006]***

- Capacity of binomial/multi-draw channel

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- Generalization to asymmetric channels
- Computation of error probabilities

Preliminaries - Channel Model Revisited

Alternative Channel Formulation

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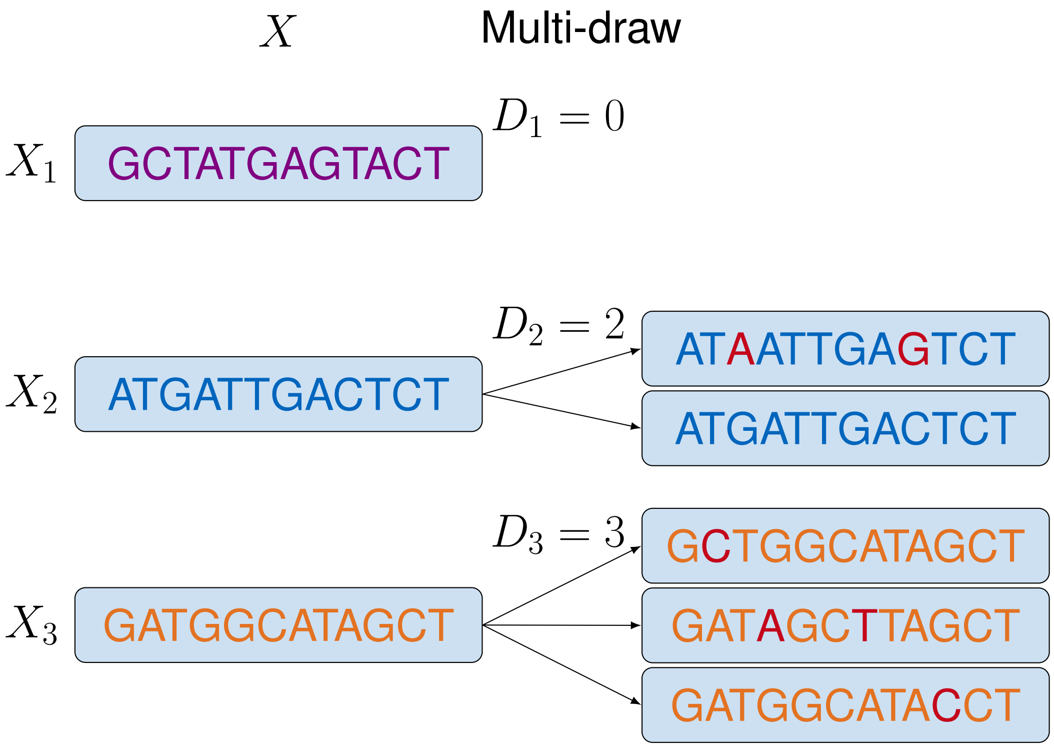
Preliminaries - Channel Model Revisited

Alternative Channel Formulation



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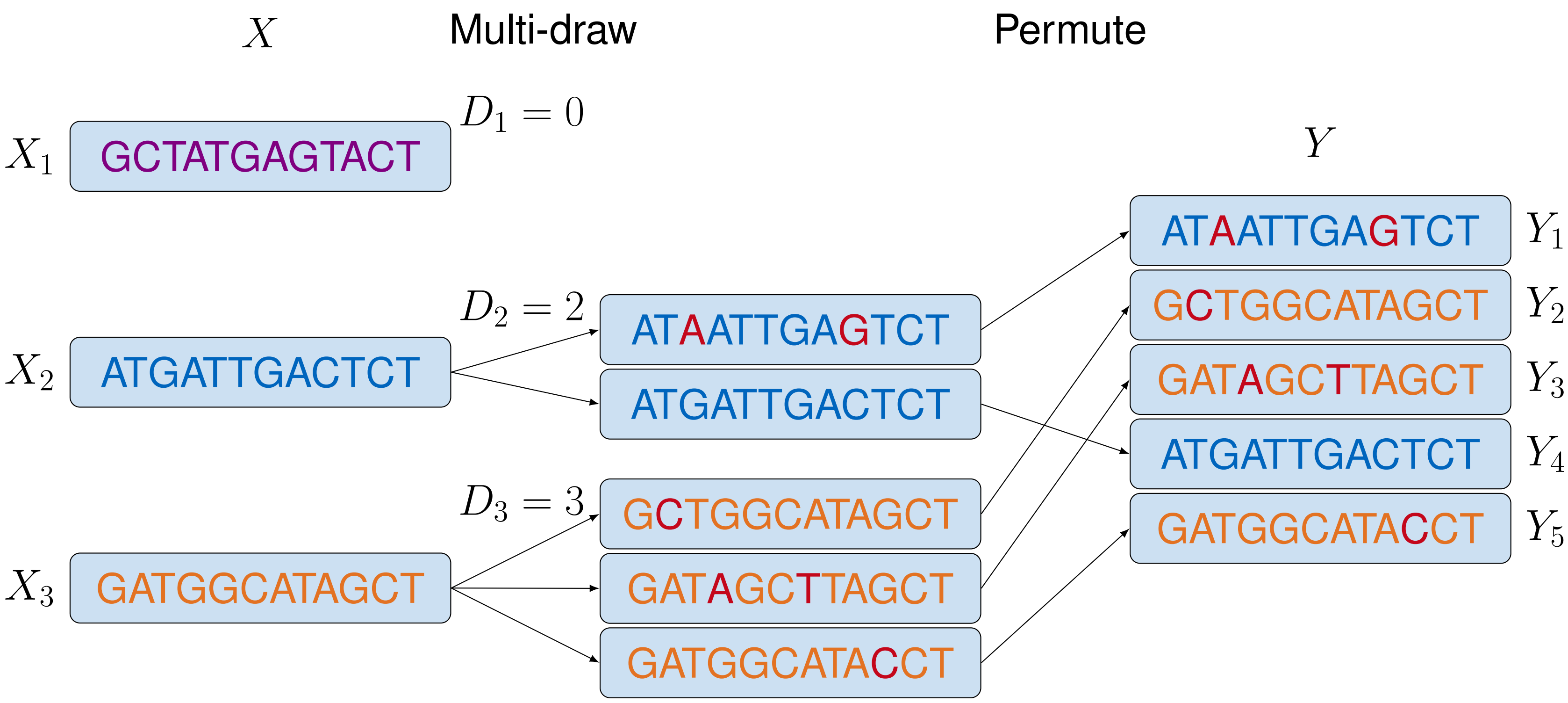

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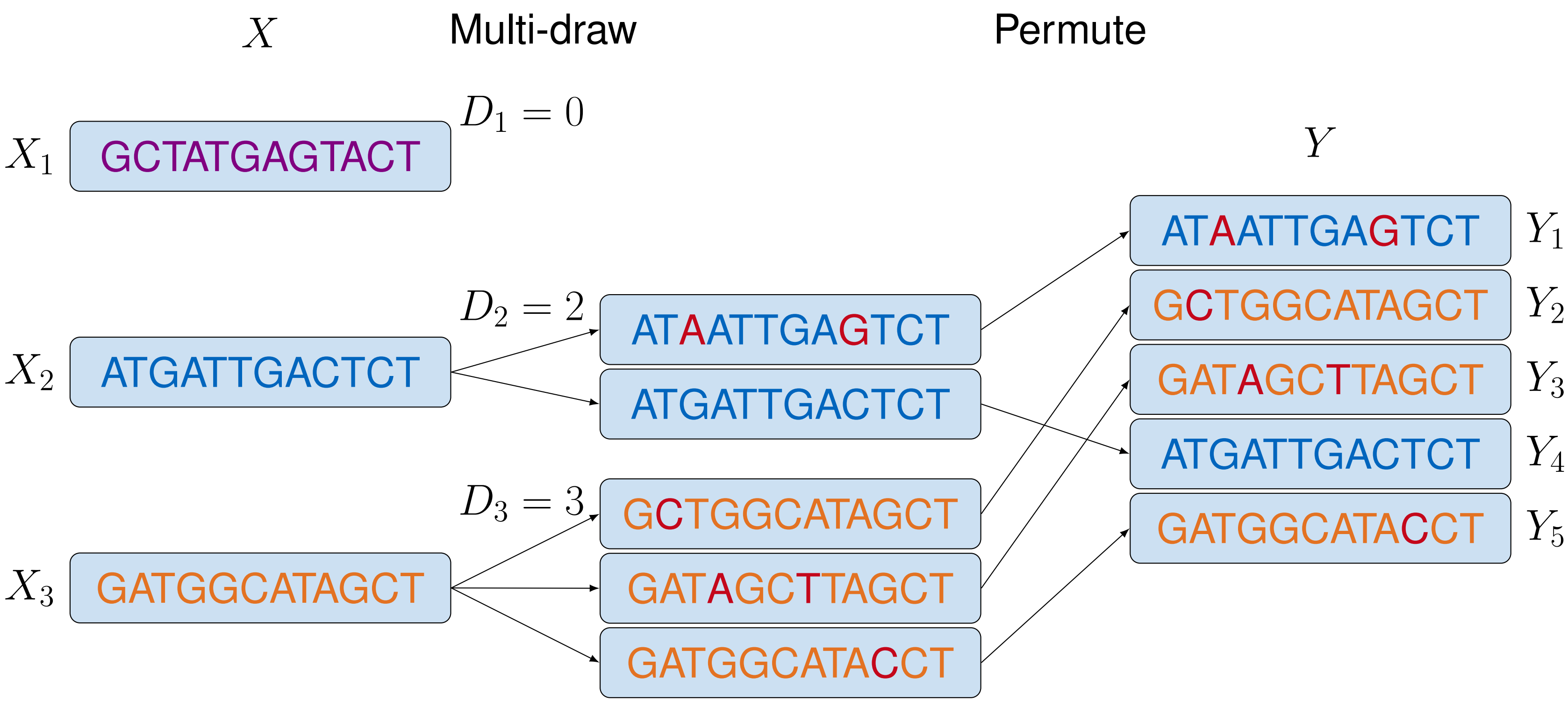

Challenges

- Draws of the multi-draw channels are random

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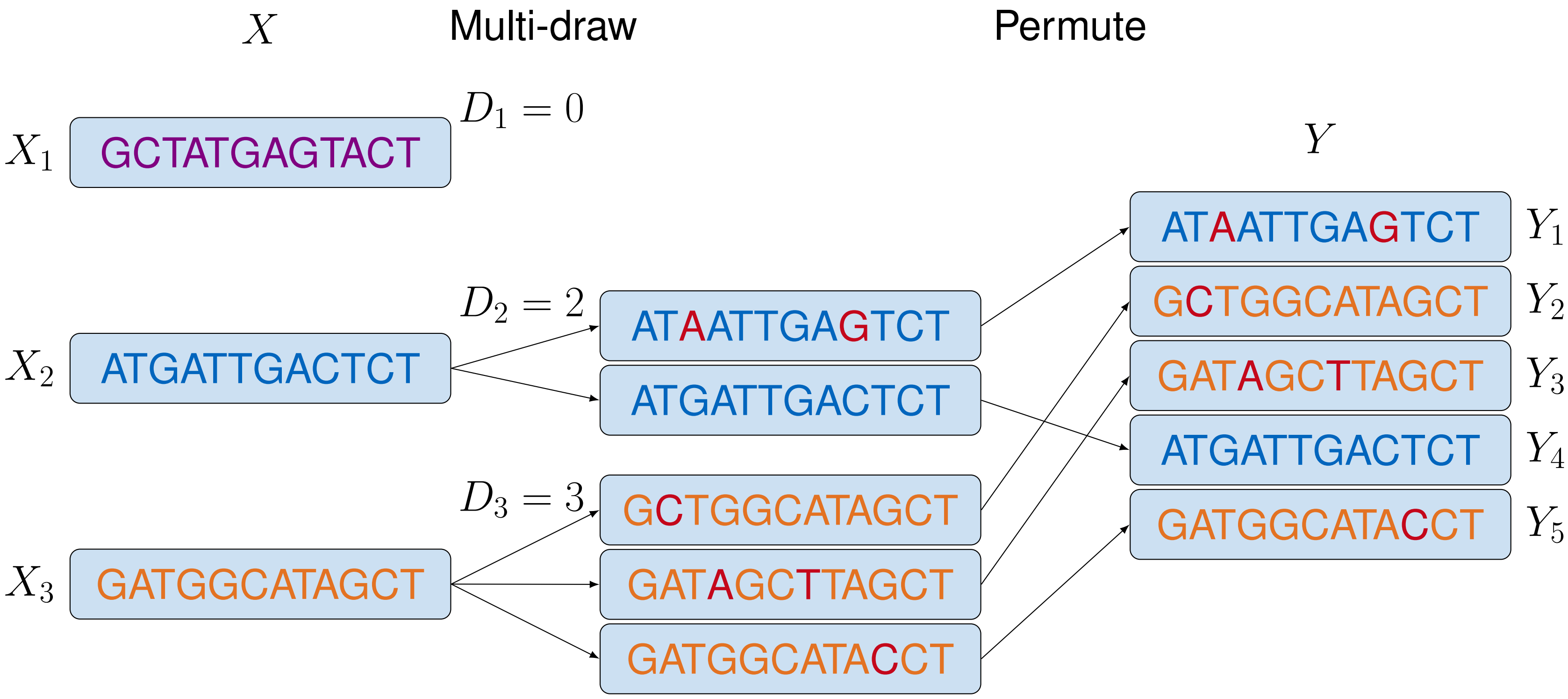
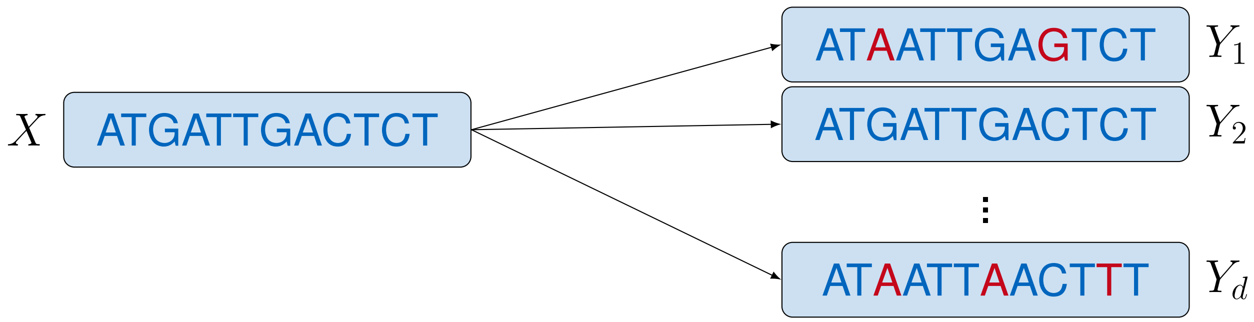


Challenges

- Draws of the multi-draw channels are random
- Permutation of the sequences

Preliminaries - Multi-draw Channel [Mitzenmacher, 2006]



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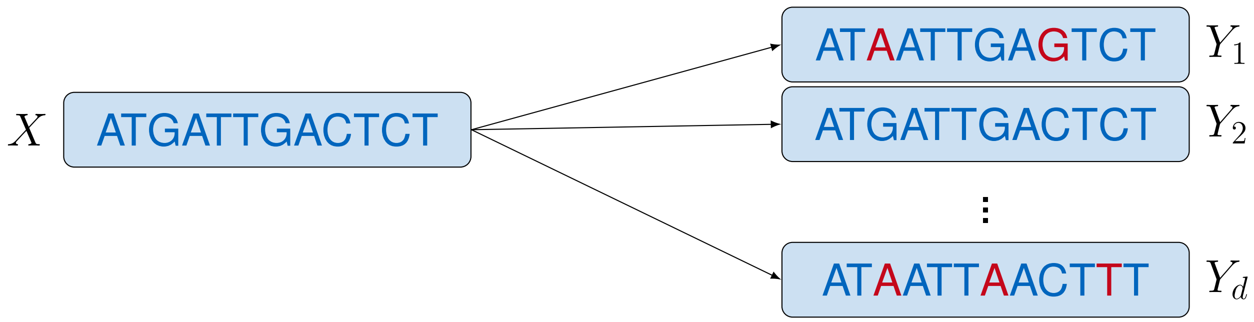

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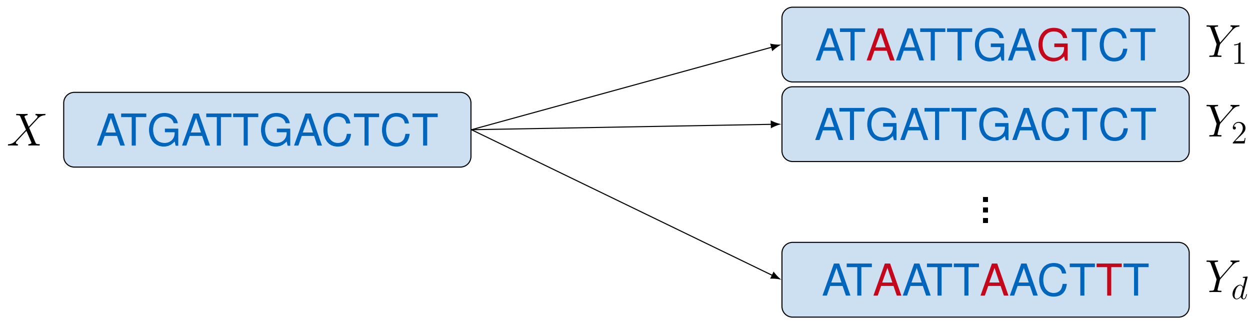
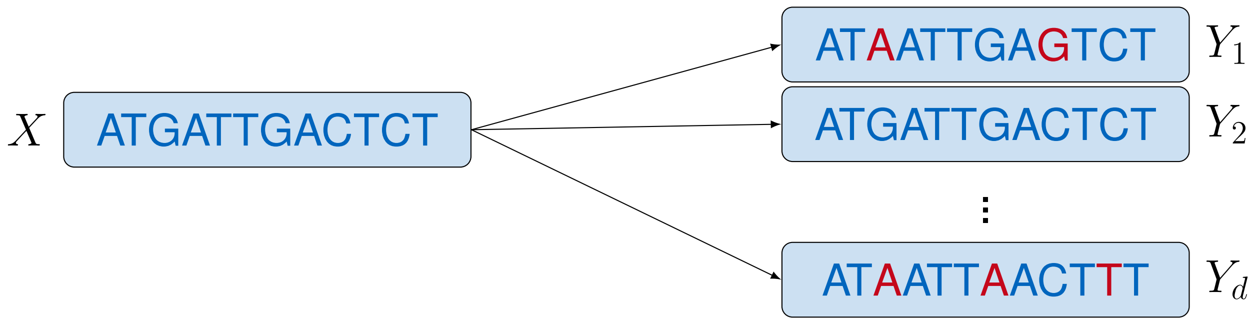
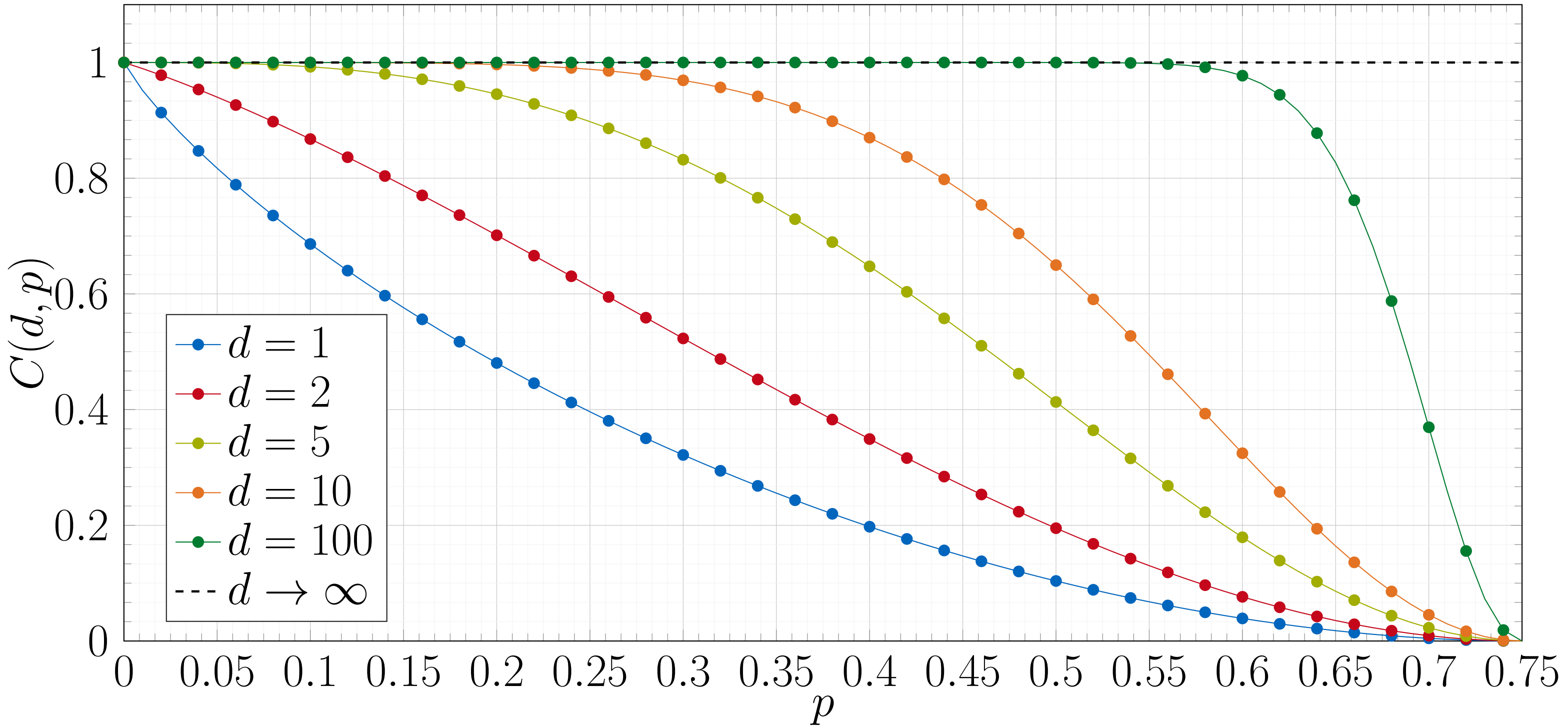


- Input: X
- Output: d output sequences $Y_1, \ldots, Y_d$
- q-ary symmetric channels: $Y_i = X + E_i$

Preliminaries - Multi-draw Channel [Mitzenmacher, 2006]



- Capacity (d draws, error probability p)



Lenz,Siegel,Wachter-Zeh,Yaakobi "*On the Capacity of DNA-based Data Storage under Substitution Errors*"

Channel Capacity

Draw Distribution

- Recall:
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Channel Capacity

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Channel Capacity

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Channel Capacity

Theorem: Channel Capacity

Given $2\beta < 1 - H_4(2p)$, the capacity is

$$C(c, \beta, p) = \sum_{d=0}^{\infty} \text{Poi}(c, d) C_{\text{Mul}}(d, p) - \beta(1 - \mathrm{e}^{-c})$$

Channel Capacity - Parameter Range

Parameter Range

$$2\beta < 1 - H_4(2p)$$

- Entropy function $H_4(p) = -(1-p)\log_4(1-p) - p\log_4\left(\frac{p}{3}\right)$

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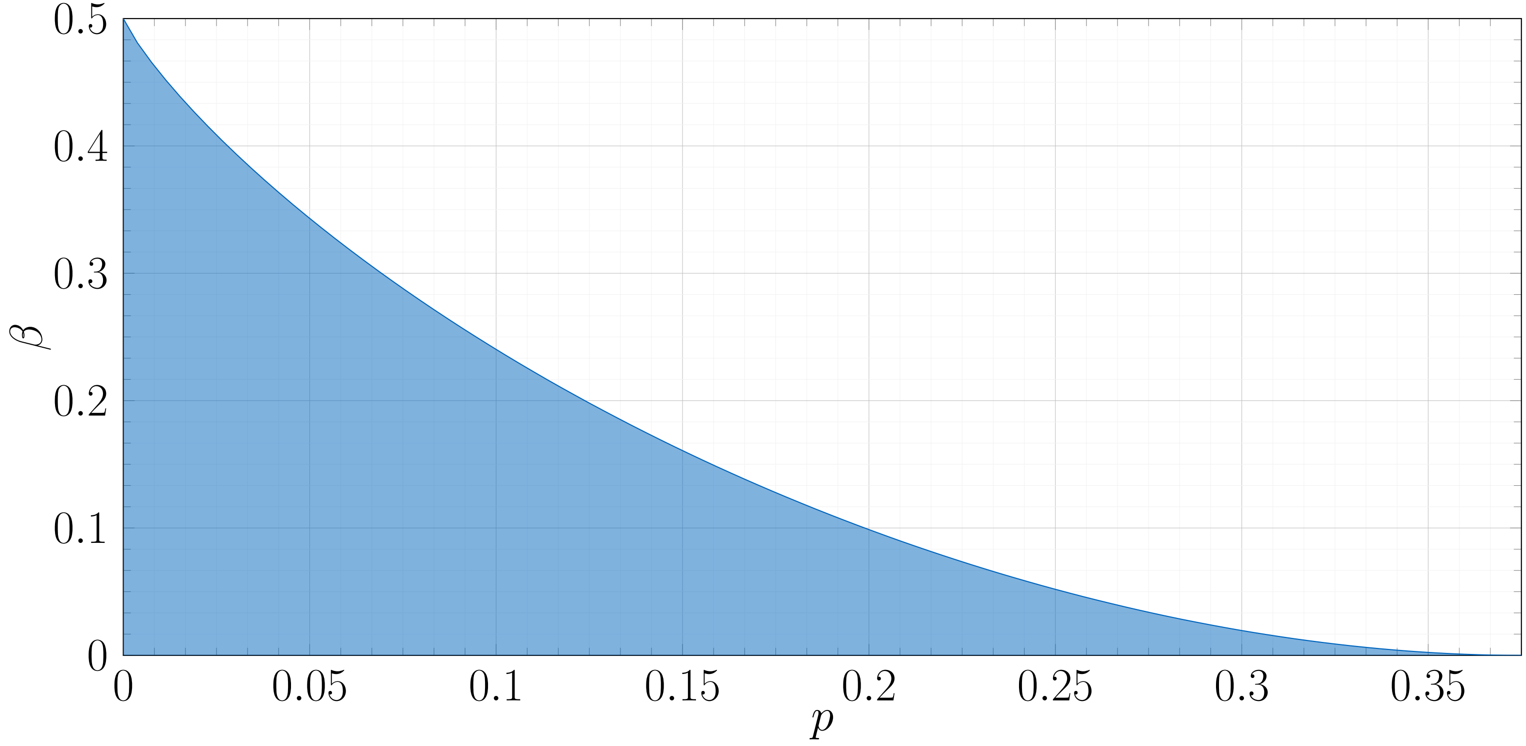

Channel Capacity - Discussion

Capacity

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Channel Capacity - Discussion

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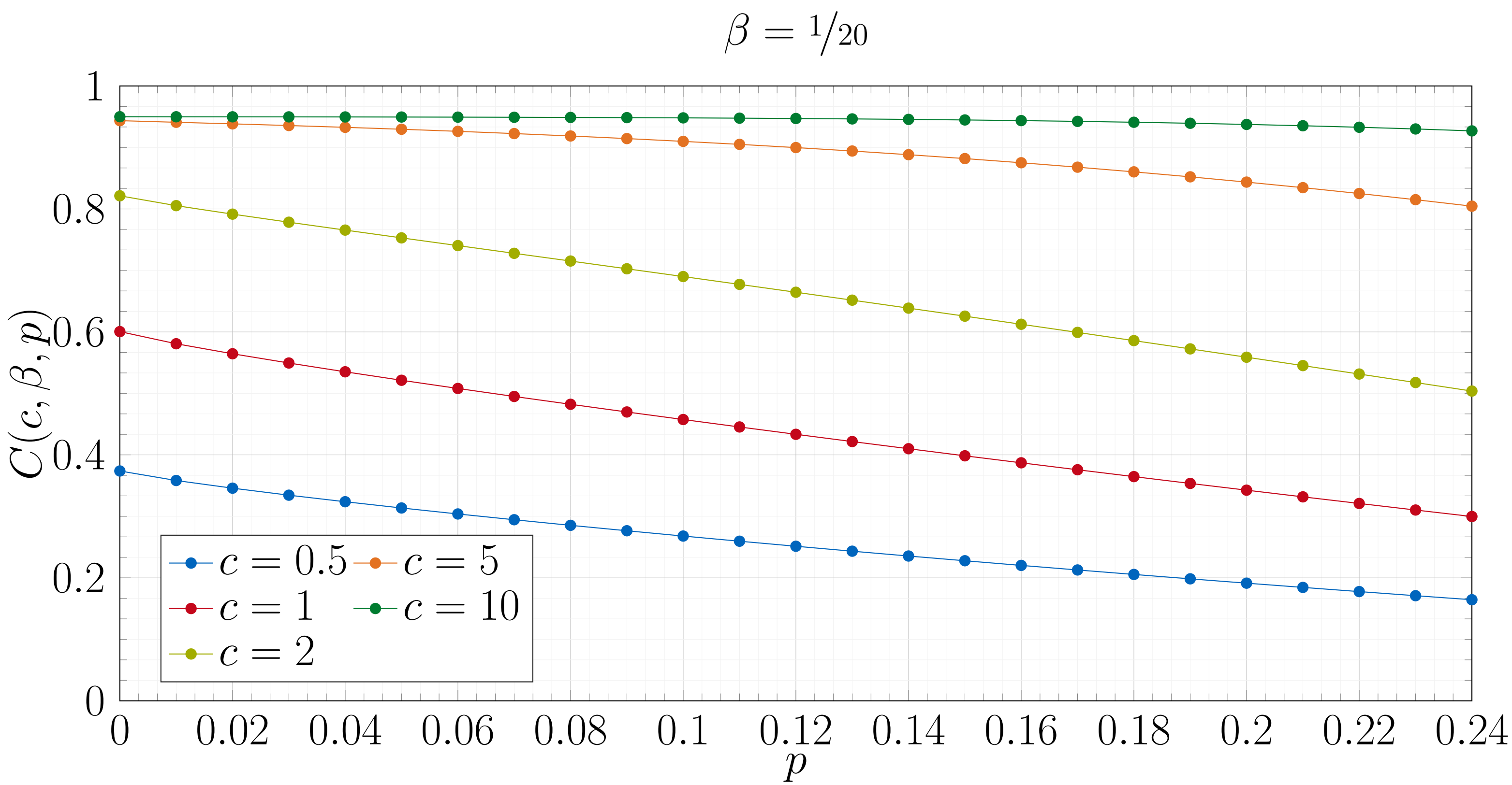

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Channel Capacity - Discussion

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$$C(c, \beta, p) = \sum_{d=0}^{\infty} \mathsf{Poi}(c, d) C_{\mathsf{Mul}}(d, p) - \beta(1 - \mathrm{e}^{-c})$$

- No errors ($p = 0$) [Heckel 2017]

Channel Capacity - Discussion

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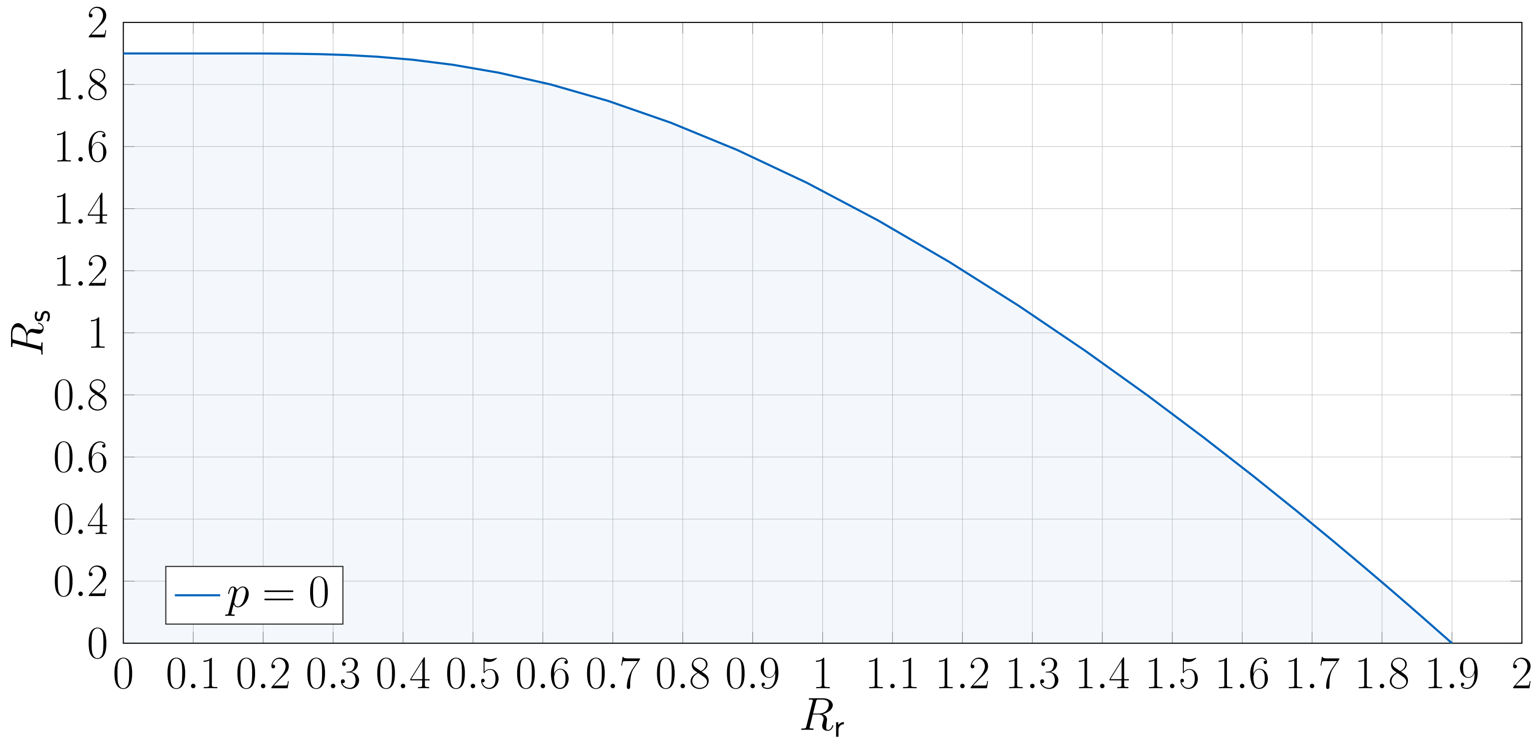

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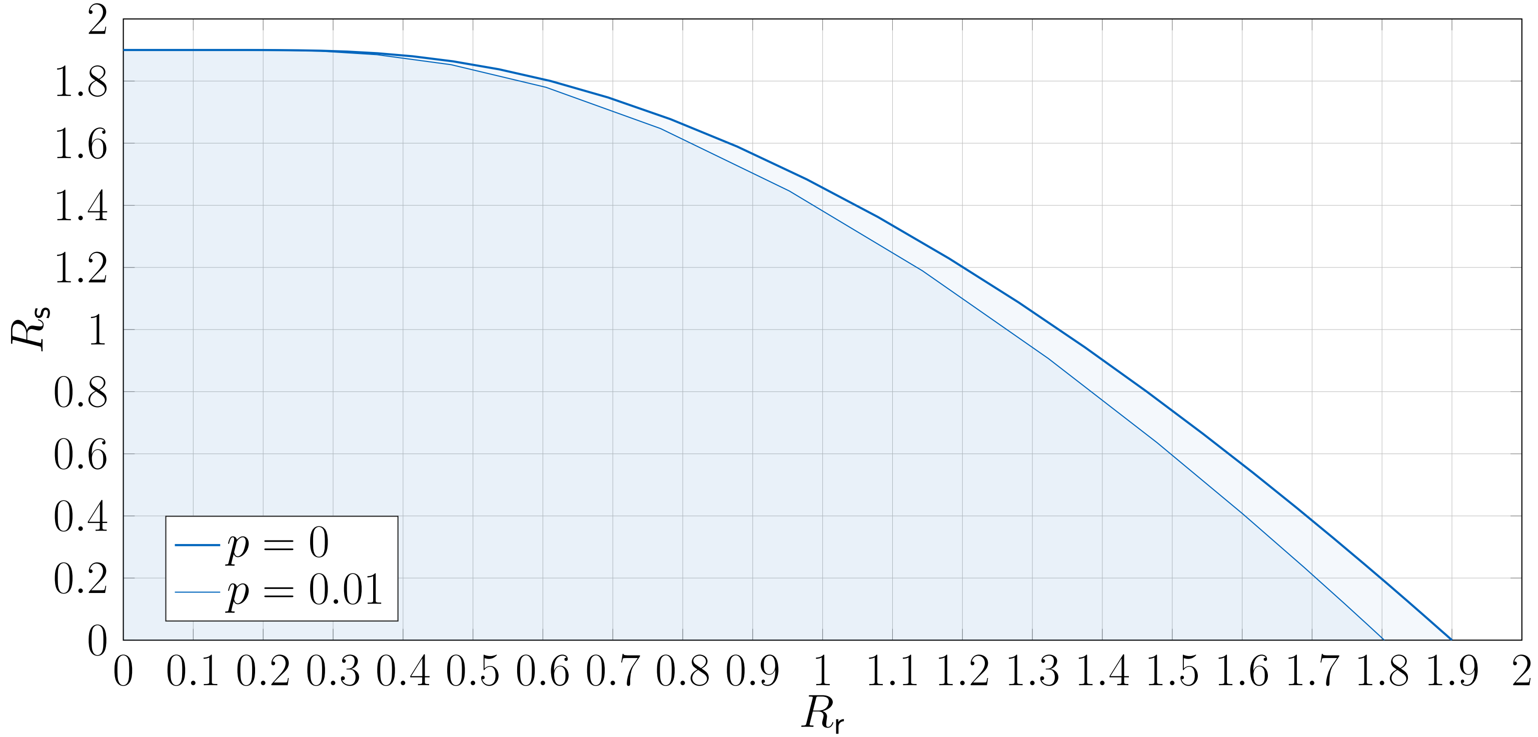

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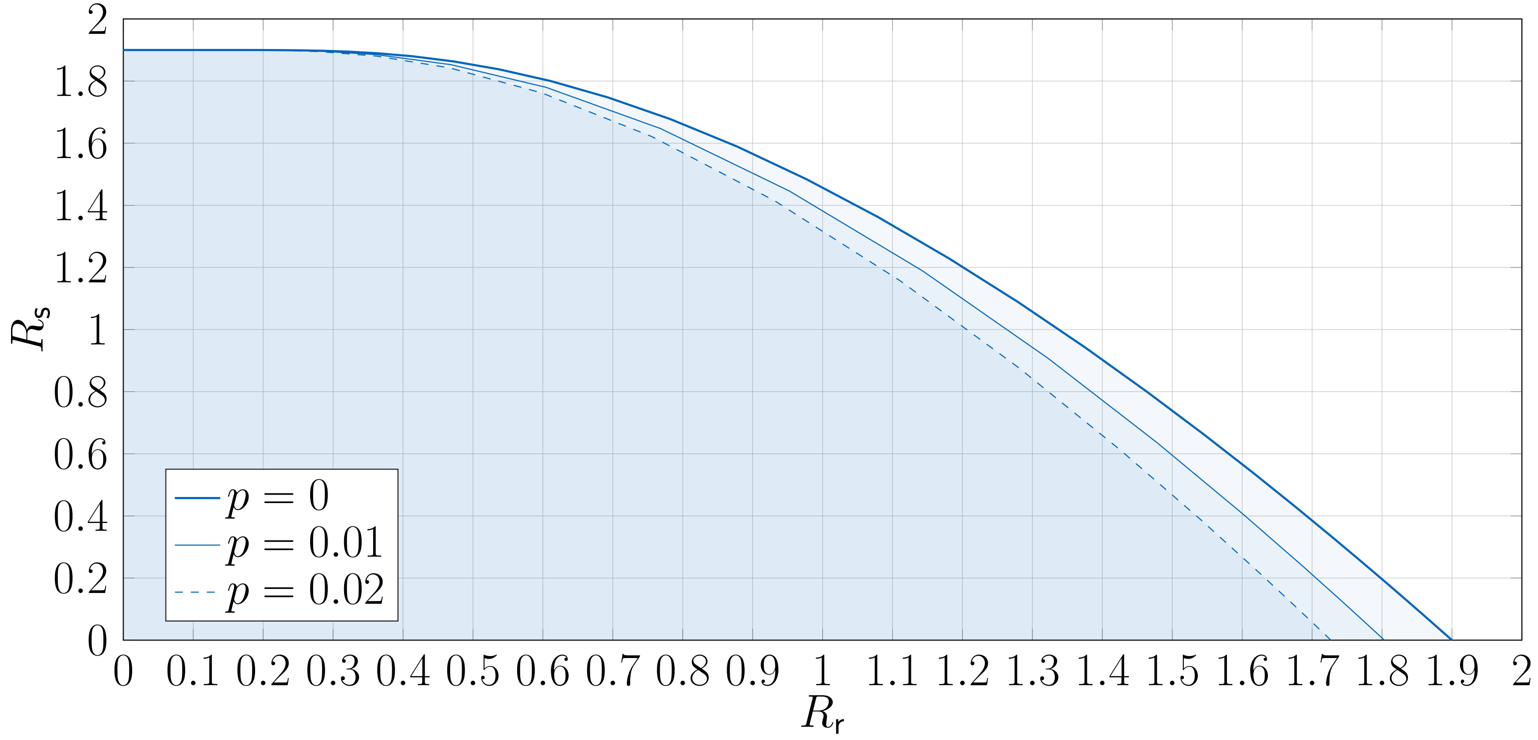

Lenz,Siegel,Wachter-Zeh,Yaakobi ”*On the Capacity of DNA-based Data Storage under Substitution Errors*”

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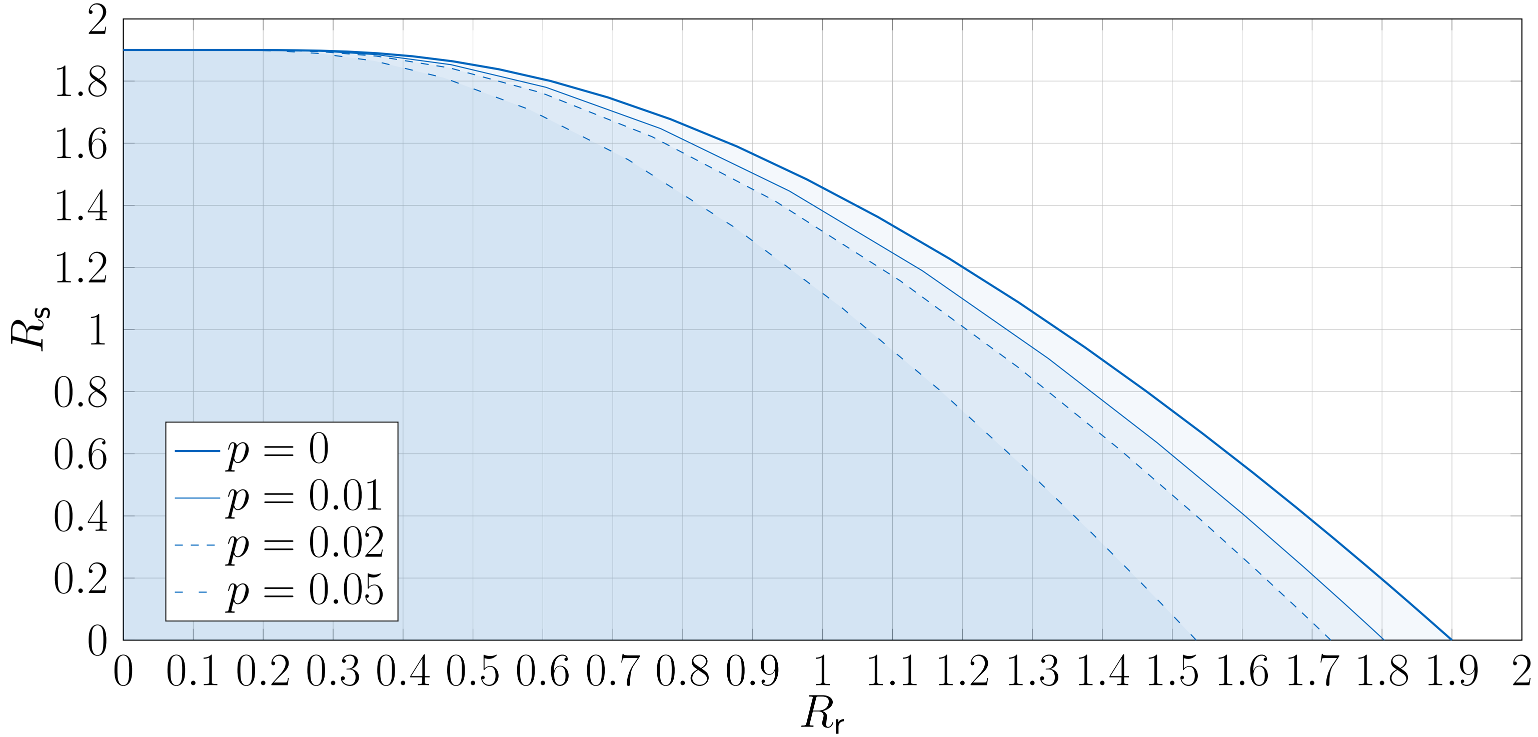

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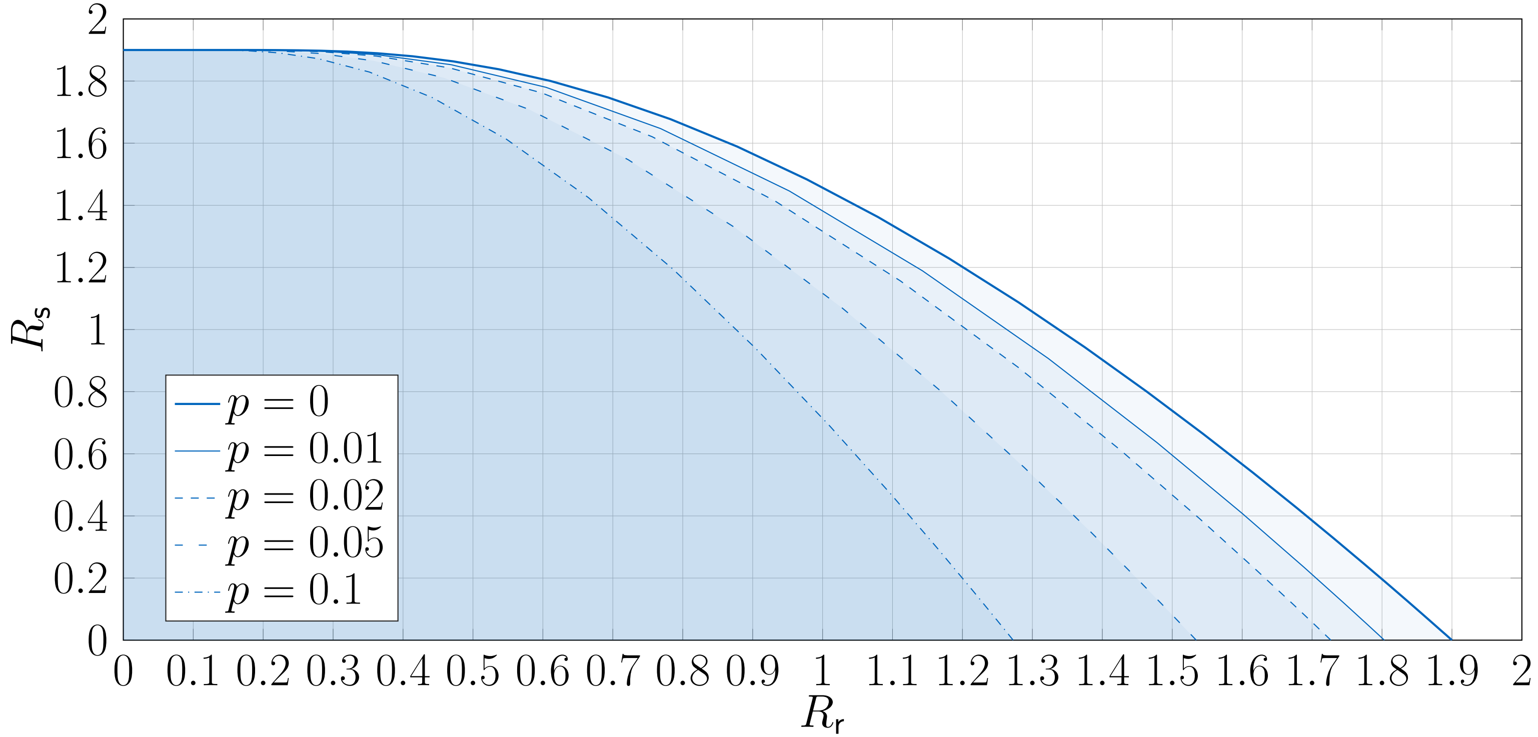

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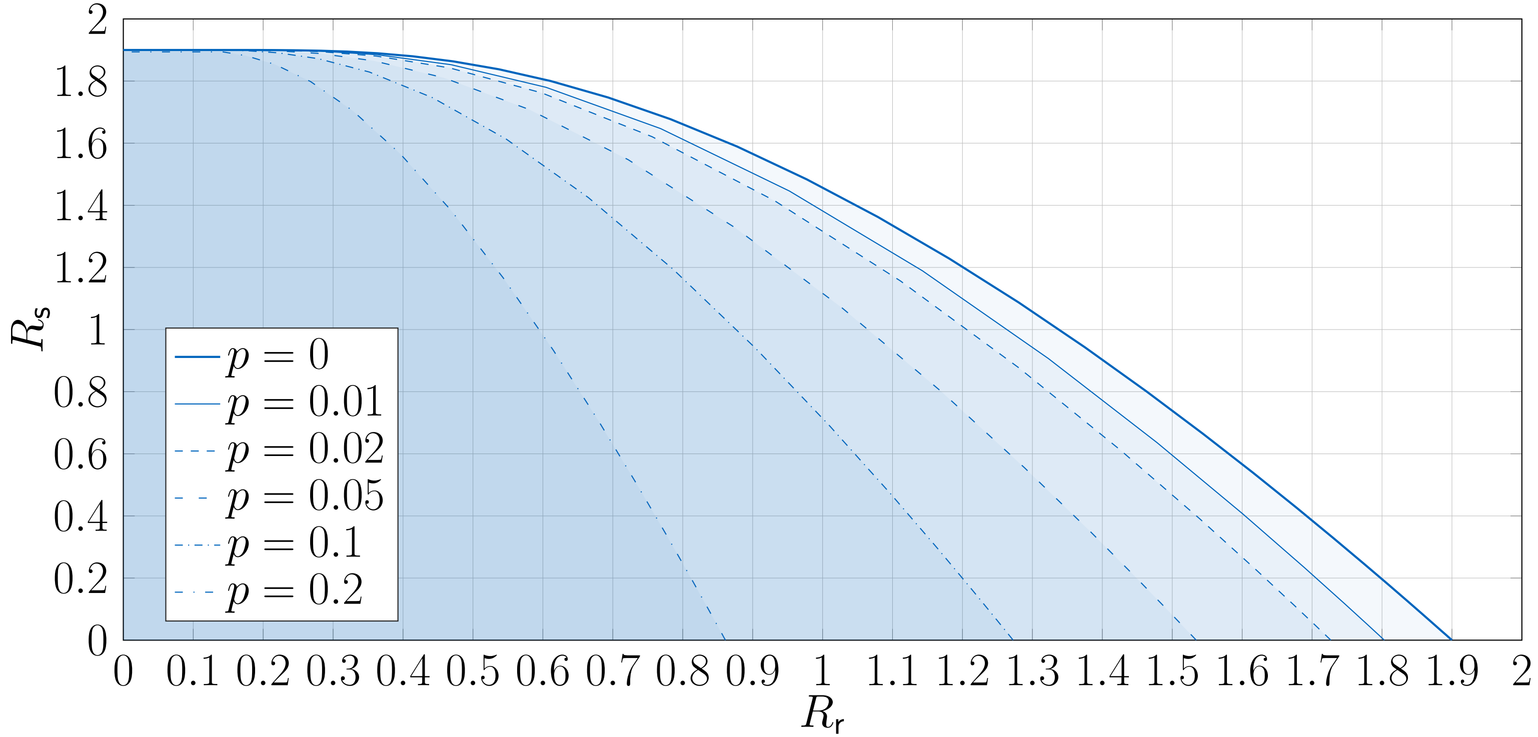

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