Random-Access LDPC Codes

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Motivation

Storage devices require **strong** error-correcting schemes.

Strong error-correcting

$\Downarrow$

Large blocks/high complexity

$\Downarrow$

Slower read access

Motivation

Extreme error events are **rare**.

Frequently, readout with modest noise

$\Downarrow$

Can we exploit this?

Motivation

Extreme error events are **rare**.

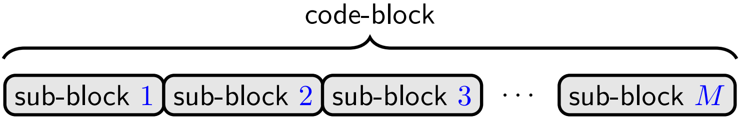
Frequently, readout with modest noise

$\Downarrow$

Can we exploit this?

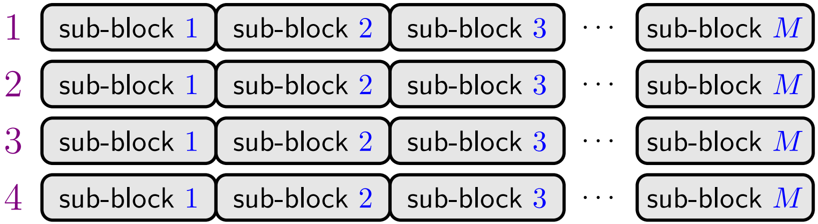
Yes: *random-access* codes.

"Random-Access" Codes

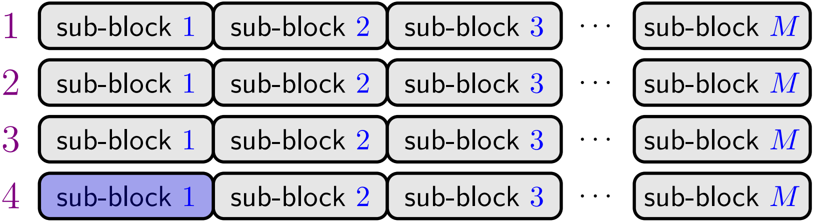


- Fast sub-block read access.
- A "safety net" code-block for increased reliability.

Random-Access Codes



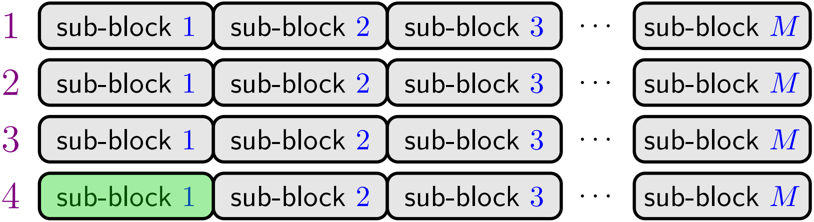
Random-Access Codes



Access sub-block

- Modest noise $\Longrightarrow$ fast access.

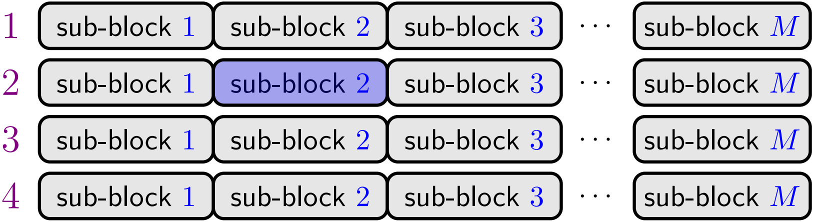
Random-Access Codes



Success

- Modest noise $\Longrightarrow$ fast access.

Random-Access Codes



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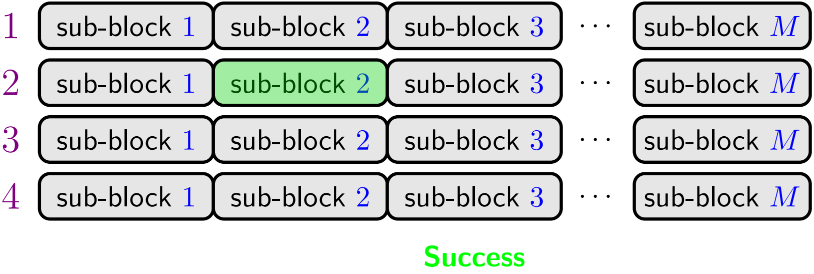
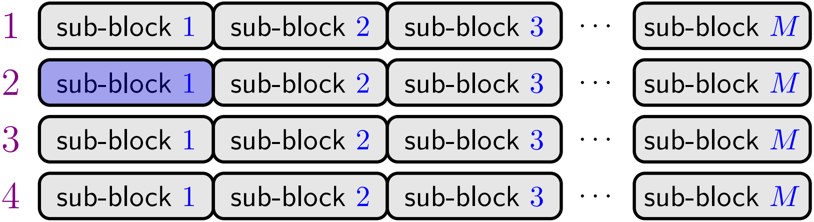
Random-Access Codes



- Modest noise $\implies$ fast access.

Random-Access Codes



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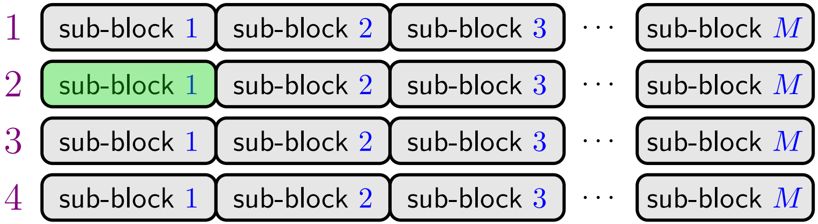

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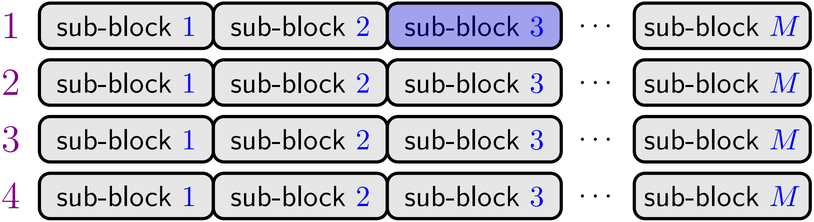


Access sub-block

- Modest noise $\implies$ fast access.

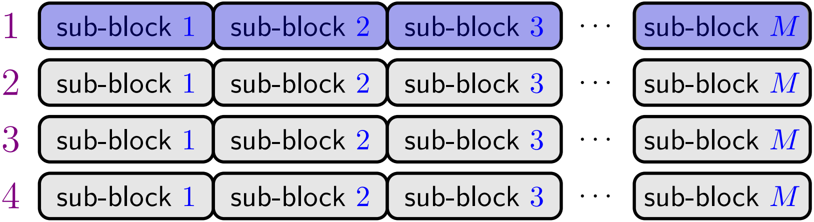
Random-Access Codes

1	sub-block 1	sub-block 2	sub-block 3	$\cdots$	sub-block M
2	sub-block 1	sub-block 2	sub-block 3	$\cdots$	sub-block M
3	sub-block 1	sub-block 2	sub-block 3	$\cdots$	sub-block M
4	sub-block 1	sub-block 2	sub-block 3	$\cdots$	sub-block M

Failure

- Modest noise $\implies$ fast access.

Random-Access Codes



Access entire code-block

- Modest noise $\Longrightarrow$ fast access.
- Extreme noise $\Longrightarrow$ safety net.

Random-Access Codes

1	sub-block 1	sub-block 2	sub-block 3	$\cdots$	sub-block M
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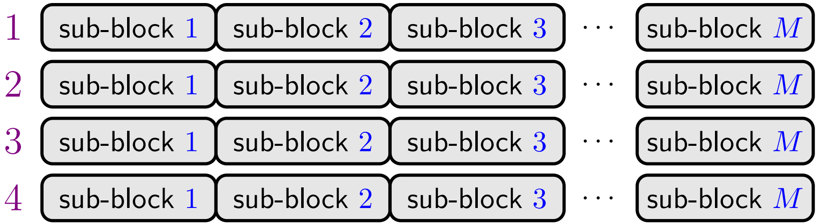
Random-Access Codes



- Modest noise $\implies$ fast access.
- Extreme noise $\implies$ safety net.

This talk: introducing random-access LDPC codes

Prior Work

- The repair problem:
  - Regenerating codes: Dimakis *et al.*, ...
  - Locally decodable codes: Tamo, Gopalan, Yekhanin, ...
- Random-access Reed-Solomon codes: Cassuto *et al.*
- Globally coupled LDPC codes: Lin, Abdel-Ghaffar, ...

Part I: LDPCL

Part I: LDPCL

LDPC constructions.

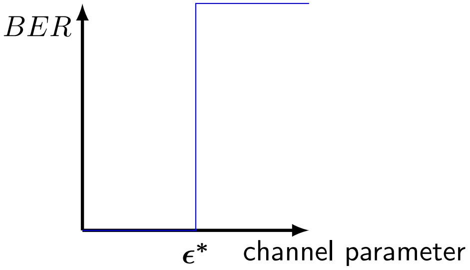
- Can we enable sub-block access?
- Can local and global codes cooperate?
- Can we prove optimality (in rate)?

Yes for all!

Belief Propagation Asymptotics – Erasures [1]

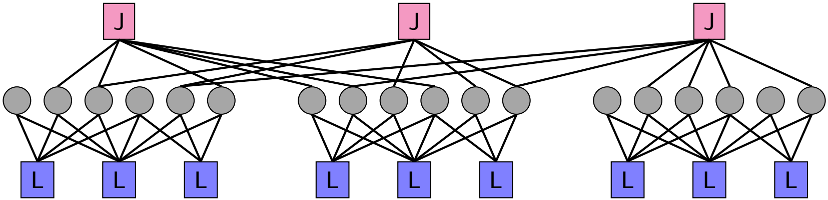
As the block length $n \to \infty$, there exists a decoding threshold ϵ^*.

- ϵ^* depends on code design.



[1] All results extend to the Gaussian channel

LDPCL Code Structure



- Local checks L : sub-block data protection.
- Joint + local checks J + L : full-block data protection.

Code Construction

- **Input:** local and global noise levels $\epsilon_L^* < \epsilon_G^*$.
- **Goal:** determine bit and check degree distributions such that:

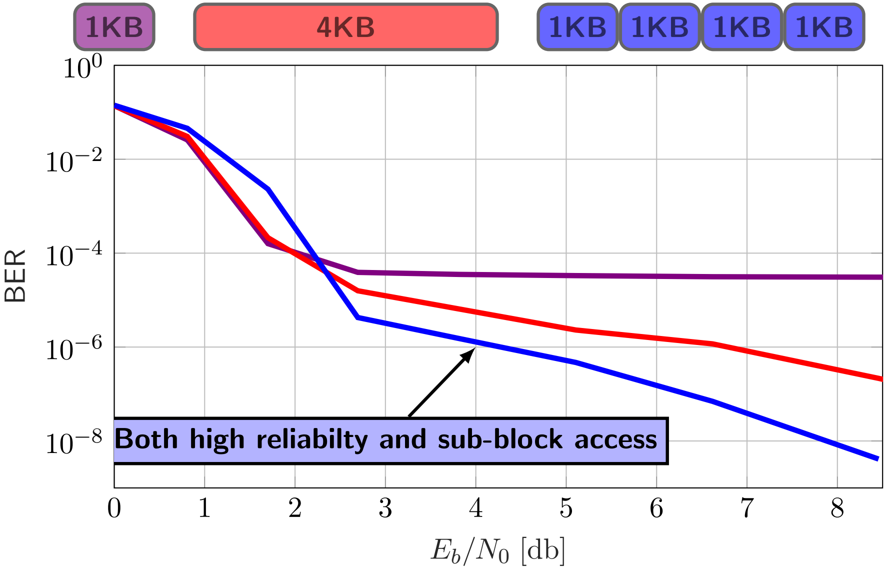
1) Data-protection ϵ_L^***:** SB 1 | SB 2 | SB 3 $\cdots$ SB M

2) Data-protection ϵ_G^***:** SB 1 | SB 2 | SB 3 $\cdots$ SB M

- Local and global codes **cooperate**.
- Approach Shannon's limit by **optimally mixing** existing codes.

[2] ER and Cassuto, "LDPC codes with local and global decoding", ISIT'18

Gaussian Channel Example: Rate $\frac{1}{2}$ Codes.



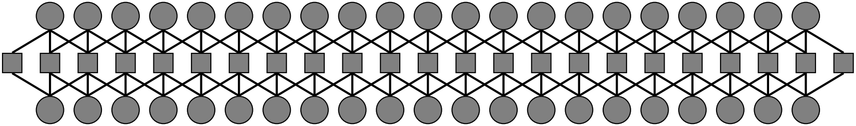
Summary of Part I

- Random-access LDPC: enjoy both worlds.
  - Proper design needed.
- More to say[1]:
  - Theory for analysis: 2-dimensional density evolution.
  - More access considerations.

[1] full arXiv: https://arxiv.org/abs/1801.03951

Part II: SC-LDPCL

- SC-LDPC codes: **very large** code-block sizes.
- Sub-block access is **valuable** for fast access.



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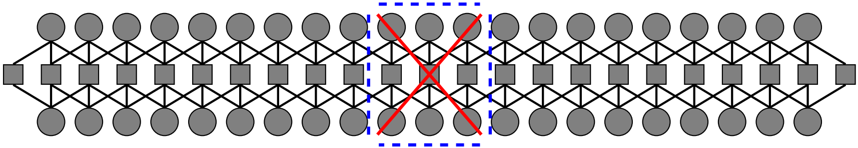
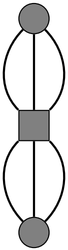


Existing SC-LDPC codes: no sub-block random access.

Protograph-Based LDPC Codes

- LDPC protograph: a small bipartite graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$.
- Lifting the **protograph** generates a **Tanner graph**.

Example: regular $(3, 6)$ protograph.



The **bi-adjacency matrix** representing the protograph:

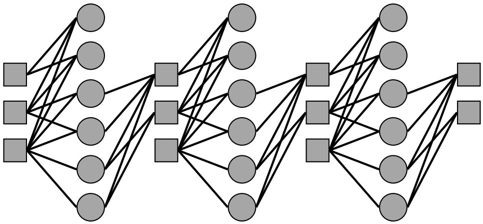
$$B = \begin{pmatrix} 3 & 3 \end{pmatrix}.$$

Protograph-Based SC-LDPC Codes:

$$H = \begin{pmatrix} B_1 & & 0 & \\ \vdots & B_1 & & \\ B_T & \vdots & \ddots & \\ & B_T & \ddots & B_1 \\ & & \ddots & \vdots \\ & 0 & & B_T \end{pmatrix}$$

Existing SC-LDPC codes: no sub-block random access

When decoding a sub-block, the bits outside the sub-block are **unknown.**



SC-LDPC protograph, VNs and CNs degrees 3 and 6

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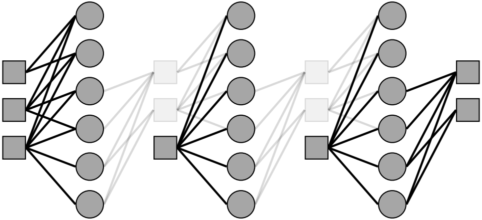

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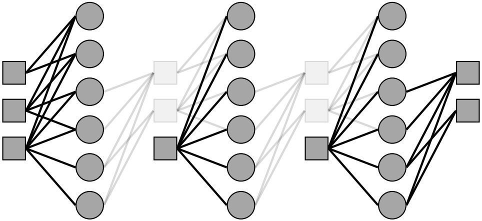
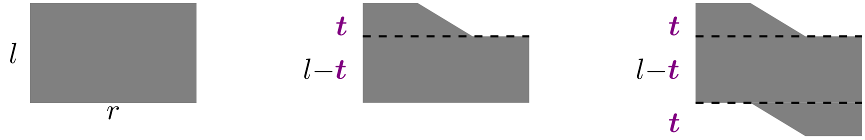


These protographs are bad: degree 1 variable nodes

Construction[1]

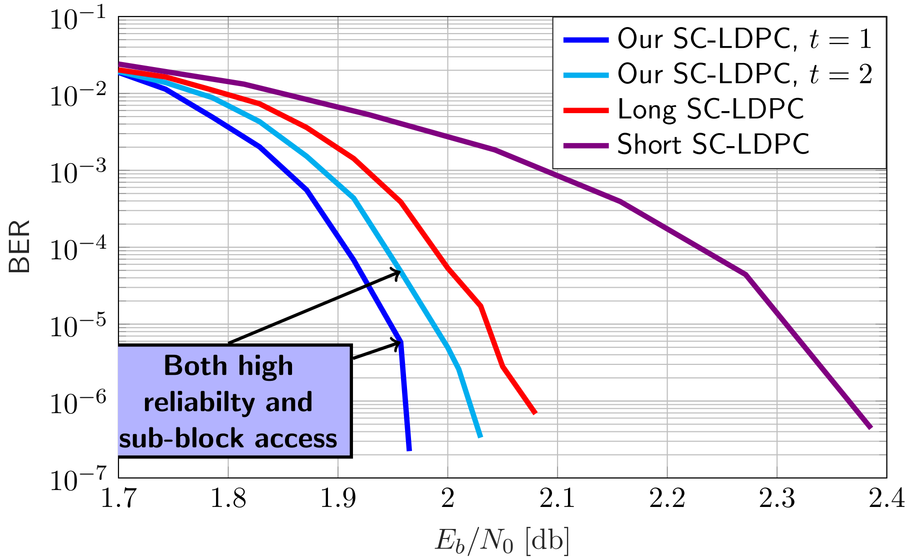


Start: B Cut: B_1 Paste: $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$

$$H = \begin{pmatrix} B_1 & & & \\ B_2 & B_1 & & \\ & B_2 & \ddots & \\ & & \ddots & B_1 \\ & & & B_2 \end{pmatrix}$$

[1] ER and Cassuto, "Spatially-coupled LDPC codes with random access", ISTC'18

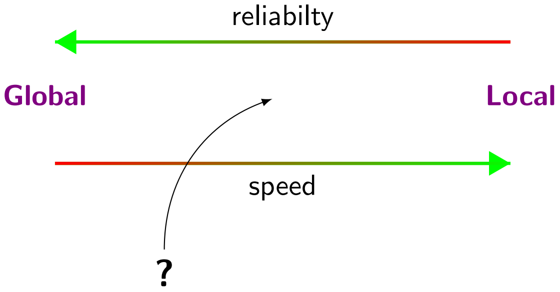
Example: SC-LDPC, VN degree 4, CN degree 8



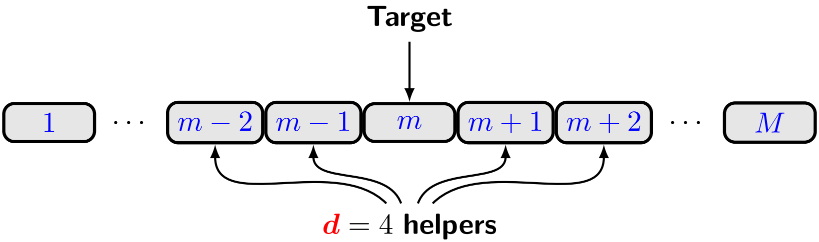
Decoding Random-Access SC-LDPC Codes

Local decoding: SB 1 | SB 2 | SB 3 $\cdots$ SB M

Global decoding: SB 1 | SB 2 | SB 3 $\cdots$ SB M



Semi-Global Decoding[1]



[1]ER and Cassuto, "On decoding random-access SC-LDPC codes", sub. to ISIT'19

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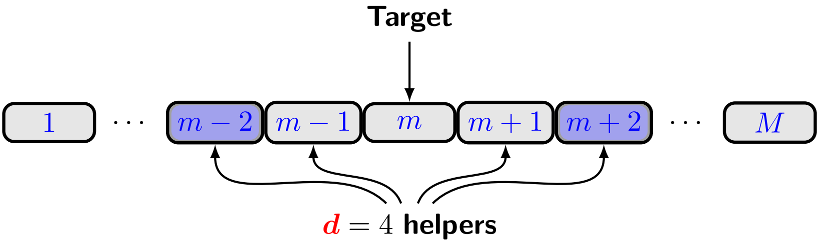

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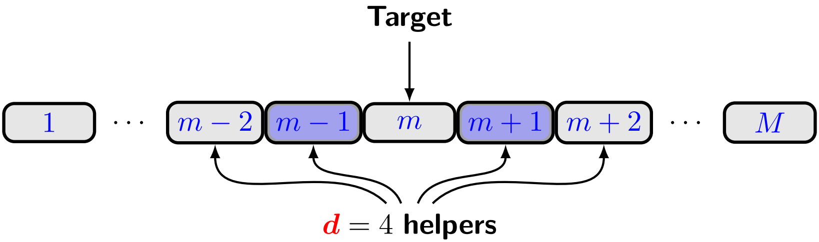

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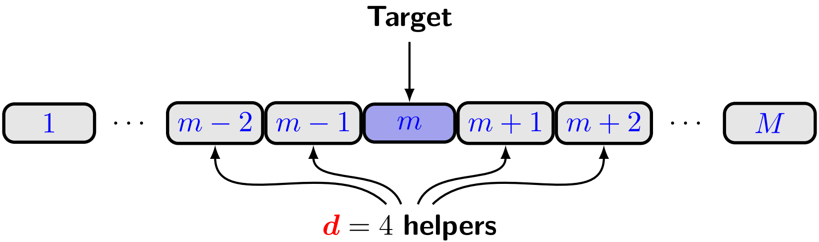

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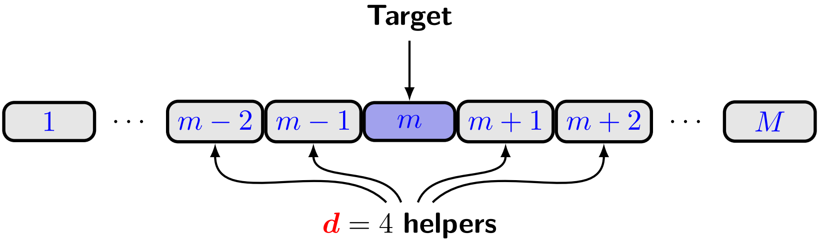
Semi-Global Decoding[1]



Semi-global decoding is more reliable than **local** decoding.

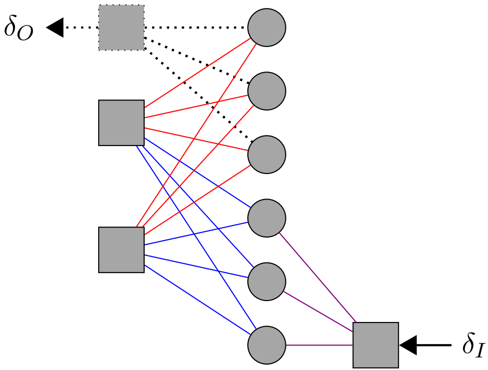
Semi-global decoding is faster than **global** decoding $d \ll M$.

[1] ER and Cassuto, "On decoding random-access SC-LDPC codes", sub. to ISIT'19

Semi-Global Density-Evolution Analysis

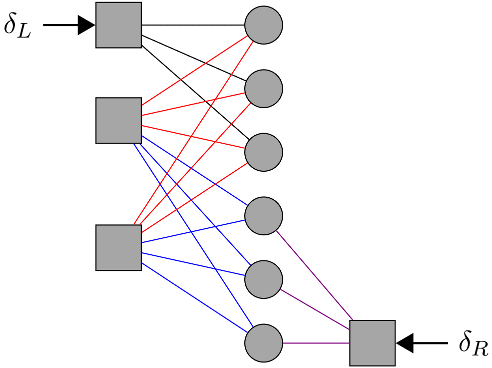
$\cdots$ $m-2$ $m-1$ m $m+1$ $m+2$ $\cdots$

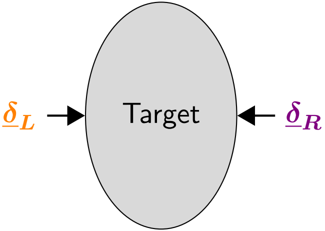
Helper



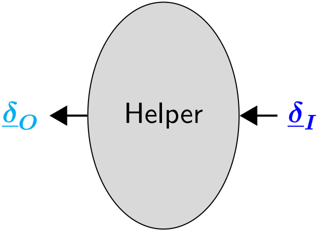
Semi-Global Density-Evolution Analysis

Target





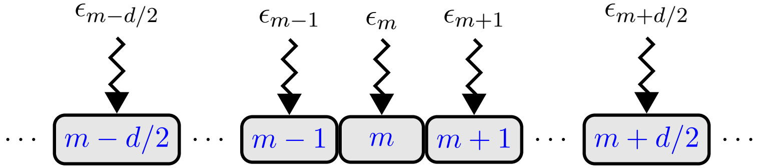
Target threshold
$\epsilon^*(\underline{\boldsymbol{\delta}}_{\boldsymbol{L}},\underline{\boldsymbol{\delta}}_{\boldsymbol{R}})$



Helper function
$\underline{\boldsymbol{\delta}}_{\boldsymbol{O}} = \Delta(\epsilon,\underline{\boldsymbol{\delta}}_{\boldsymbol{I}})$

The Sub-Block Varying Channel

Different sub-blocks suffer different channel parameters[3].



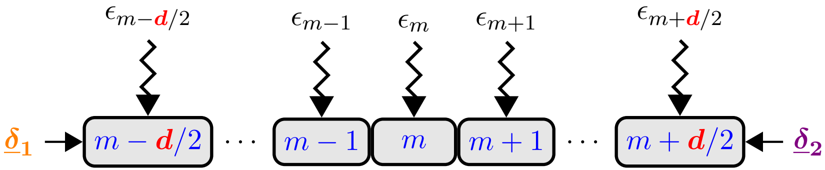
Even if the **target** suffers **low SNR**, maybe the **helpers** have **high SNR**.

$$\Downarrow$$

Semi-global decoding

[3] McEliece and Stark, "Channels with block interference"

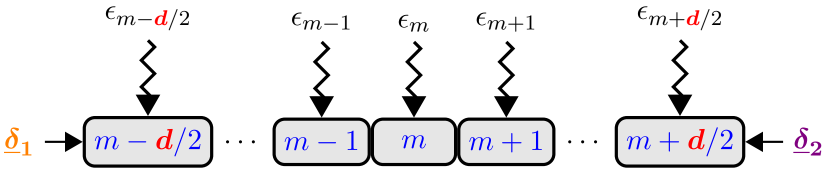
Assumptions



- Erasure rates are i.i.d. random variables.
- $d \ll M$.
- $\underline{\delta}_1$ and $\underline{\delta}_2$: erasure rates coming from the **left** and **right**.

$p_d(\underline{\delta}_1, \underline{\delta}_2)$: semi-global success probability.

Assumptions



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$p_d(1, 1)$: semi-global success probability.

Target threshold — Helper function

$\epsilon^*(\cdot,\cdot)$ — $\Delta(\cdot,\cdot)$

$p_{\boldsymbol{d}}(\underline{\boldsymbol{\delta}}_{\mathbf{1}}, \underline{\boldsymbol{\delta}}_{\mathbf{2}})$

semi-global success probability

Theorem

For every even $\boldsymbol{d} \geq 0$

$$p_{\boldsymbol{d}}\left(\underline{\boldsymbol{\delta}}_{\mathbf{1}}, \underline{\boldsymbol{\delta}}_{\mathbf{2}}\right) = \mathbb{E}\left[p_{\boldsymbol{d}-2}\left(\Delta\left(E, \underline{\boldsymbol{\delta}}_{\mathbf{1}}\right), \Delta\left(E', \underline{\boldsymbol{\delta}}_{\mathbf{2}}\right)\right)\right], \ \boldsymbol{d} > 0,$$
$$p_0(\underline{\boldsymbol{\delta}}_{\mathbf{1}}, \underline{\boldsymbol{\delta}}_{\mathbf{2}}) = \Pr\left(E < \epsilon^*(\underline{\boldsymbol{\delta}}_{\mathbf{1}}, \underline{\boldsymbol{\delta}}_{\mathbf{2}})\right),$$

E, E': *channel parameter random variables.*

Proposition

For every even $d \geq 2$, $p_d(\underline{1},\underline{1})$ is lower bounded by

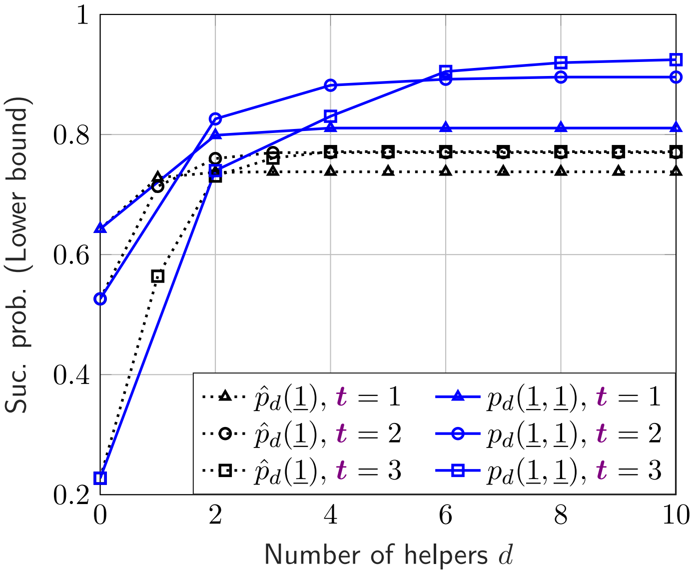
$$P_L^2 \cdot p_{d-2}\left(\underline{0},\underline{0}\right) + 2P_L(1-P_L)p_{d-2}\left(\underline{1},\underline{0}\right) + \left(1-P_L\right)^2 p_{d-2}\left(\underline{1},\underline{1}\right),$$

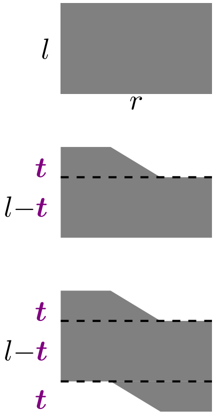
and

$$P_L + 2(P_S - P_L)(\hat{p}_{\frac{d}{2}-1}(\underline{1})) + (P_D - 2P_S + P_L)\left(\hat{p}_{\frac{d}{2}-1}(\underline{1})\right)^2,$$

- $P_L \triangleq \Pr\left(E < \epsilon^*(\underline{1},\underline{1})\right)$.
- $P_S \triangleq \Pr\left(E < \epsilon^*(\underline{0},\underline{1})\right)$.
- $P_D \triangleq \Pr\left(E < \epsilon^*(\underline{0},\underline{0})\right)$.
- $\hat{p}_d(\underline{1})$: all helpers are at **one side** of the target.

Ex.: VN deg = 5, CN deg = 12, BEC(E), $E \sim U[0, 0.4]$





Summary of Part II

- Design and analysis of **random-access SC-LDPC** codes.
  - Driven by practical problems.
- **Semi-global** decoding is a **middle-way** decoding mode.
  - Fits well to the sub-block varying channel.
  - New analysis needed.
- Future work: finite block length.

Summary of Part II

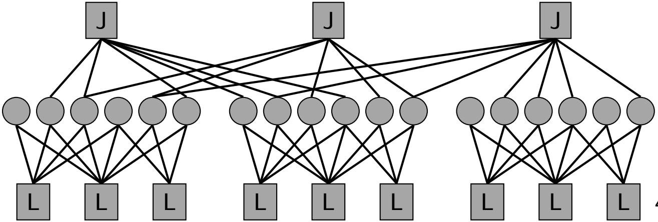
Thank You!

Backup slides

LDPC Codes with Locality (LDPCL)

Every LDPC code can be described by a **Tanner graph**.

We define LDPCL codes via **multi-block Tanner graphs**.



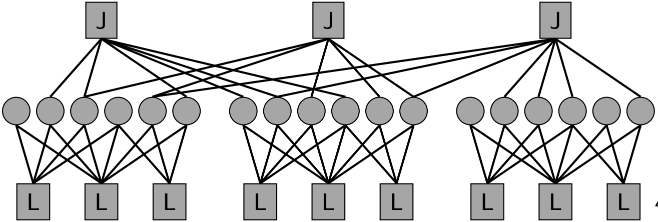
[4]

[4]ER and Cassuto, "LDPC codes with local and global decoding", ISIT'18

LDPC Codes with Locality (LDPCL)

Local checks ('L'): connected to variable nodes in the same sub-block.

Joint checks ('J'): increase data reliability in case of sub-block errors.



[4]

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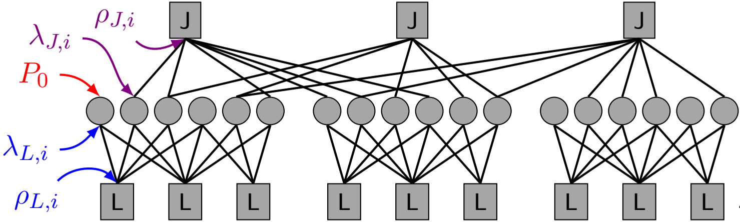
LDPC Codes with Locality (LDPCL)

We characterize multi-block Tanner graphs via **degree distributions**.

- **Local** degree distributions $\{\lambda_{L,i}, \rho_{L,i}\}_{i\geq 2}$.
- **Joint** degree distributions $\{\lambda_{J,i}, \rho_{J,i}\}_{i\geq 2}$.
- P_0 – fraction of **jointly-unconnected** variable nodes.



[4]

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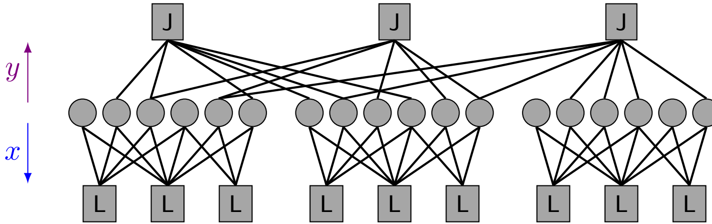
LDPC Codes with Locality (LDPCL)

Theory of analysis:

- Due to graph structure, 2 densities should be tracked: x, y.
- During global decoding, local and joint cooperate
  - Joint alone is not enough.



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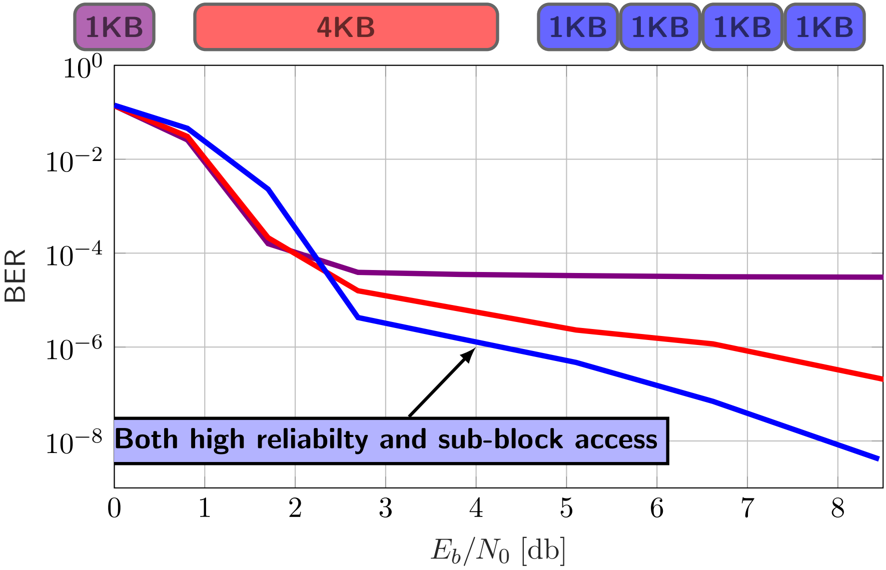
Example: rate $\frac{1}{2}$ codes.



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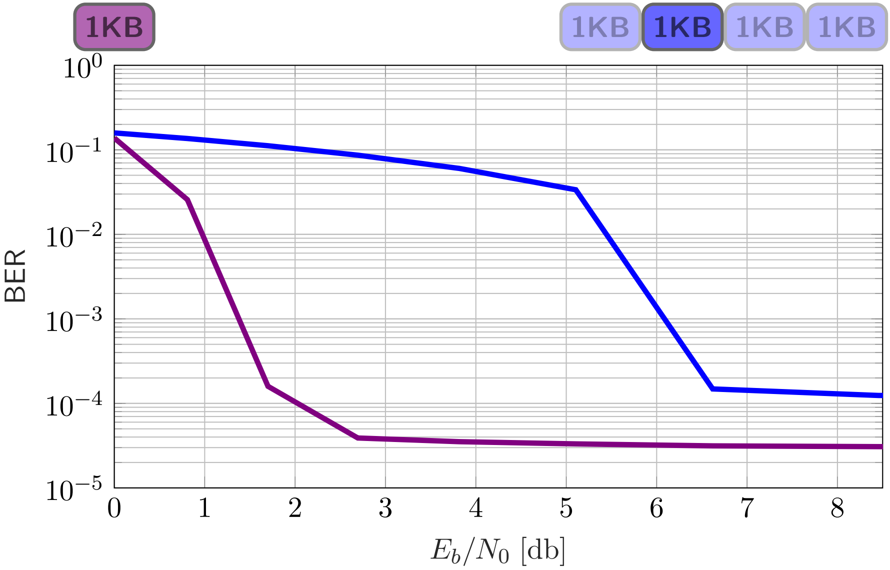


(l, r, t) SC-LDPCL Construction

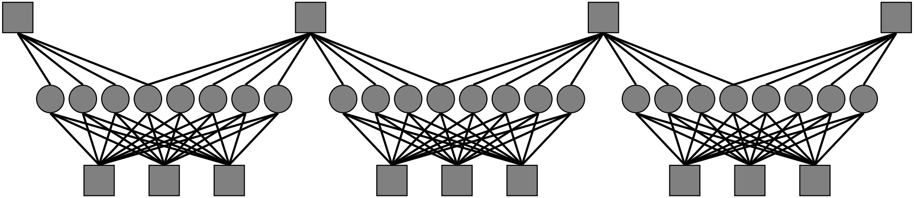
1. Let $\boldsymbol{t} \in \{1, 2, \ldots, l-2\}$, and let A_1 be a $\boldsymbol{t} \times r$ matrix given by

$$A_1 = \begin{pmatrix} \underline{1} & & & & & \underline{0} \\ \underline{1} & \underline{1} & & \text{\huge 0} & & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & & & \underline{0} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \underline{1} & \underline{1} & \underline{1} & \cdots & \underline{1} & \underline{0} \end{pmatrix},$$

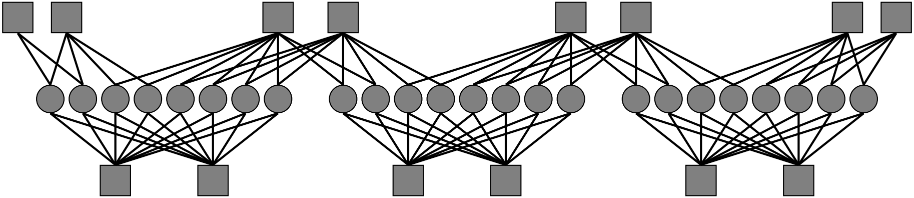
2. $A_2 = 1^{(l-t)\times r}$: an all-ones matrix.
3. $B_1 = \big(A_1; A_2\big)$, and $B_2 = 1^{l\times r} - B_1$.

Examples

$(l=4, r=8, \boldsymbol{t}=1)$

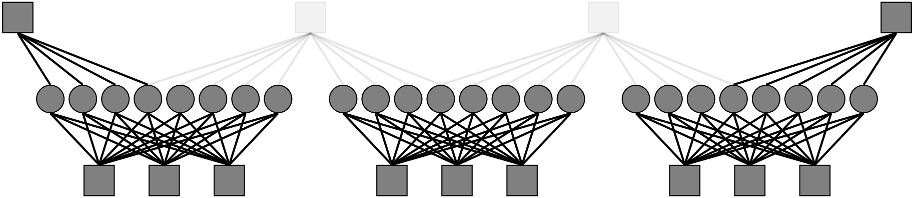


$(l=4, r=8, \boldsymbol{t}=2)$

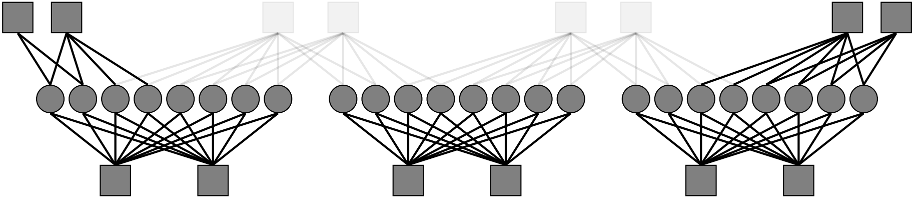


Examples

$$(l=4, r=8, \boldsymbol{t}=1)$$

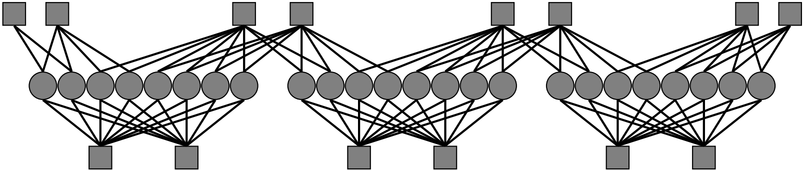


$$(l=4, r=8, \boldsymbol{t}=2)$$



Semi-Global Decoding

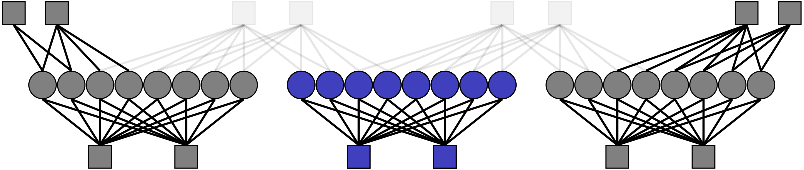
Observation:
If we successfully decode a sub-block, its neighbors become more reliable.



Semi-Global Decoding

Observation:

If we successfully decode a sub-block, its neighbors become more reliable.

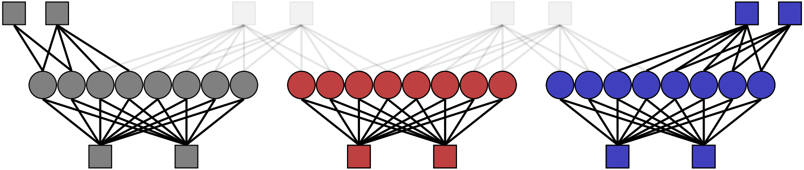


Access target sub-block

Semi-Global Decoding

Observation:

If we successfully decode a sub-block, its neighbors become more reliable.

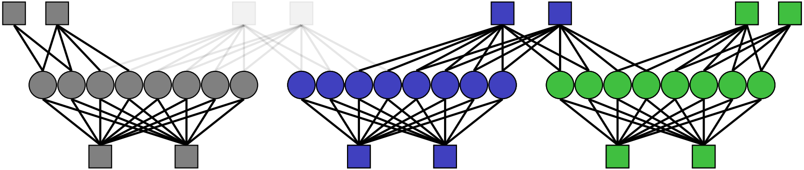


Failure: access helper sub-block

Semi-Global Decoding

Observation:

If we successfully decode a sub-block, its neighbors become more reliable.

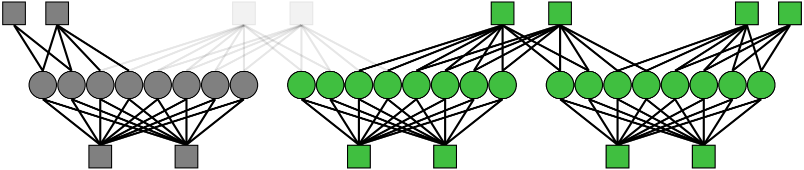


Helper sucess: access target sub-block again

Semi-Global Decoding

Observation:

If we successfully decode a sub-block, its neighbors become more reliable.



Target sucess!