Multi-Dimensional Spatially-Coupled Code Design with Improved Cycle Properties

Homa Esfahanizadeh, Ahmed Hareedy, and Lara Dolecek
Department of Electrical and Computer Engineering, University of California, Los Angeles, USA
hesfahanizadeh@ucla.edu, ahareedy@ucla.edu, and dolecek@ee.ucla.edu

***Abstract*—A spatially-coupled (SC) code is constructed by coupling disjoint block codes into a single coupled chain. By connecting (coupling) several SC codes, multi-dimensional SC (MD-SC) codes are constructed. In this work, we present a systematic framework for constructing MD-SC codes with notably better cycle properties than the 1D-SC counterparts. Compared to the 1D-SC codes, our MD-SC codes are demonstrated to have up to 85% reduction in the population of the smallest cycle, and up to 3.8 orders of magnitude BER improvement in the early error floor region. The results of this work can be particularly beneficial in data storage systems, e.g., 2D magnetic recording and 3D Flash systems, as high-performance MD-SC codes are robust against various channel impairments and non-uniformity.**

I. Introduction and Preliminaries

Spatially-coupled (SC) codes are a family of graph-based codes that have attracted significant attention thanks to their capacity approaching performance [2]. Multi-dimensional SC (MD-SC) codes are constructed by coupling several SC codes together via rewiring the existing connections or by adding extra variable nodes (VNs) or check nodes (CNs) [3], [4]. Previous works on MD-SC codes, while promising, have some limitations. In particular, they either consider random constructions, are limited to specific topologies, or consider the asymptotic behaviour, e.g. [3]–[6].

MD-SC codes are more robust against burst erasures and channel impairments, e.g., multi-dimensional inter symbol interference (ISI) that occur in 2D magnetic recording and 3D Flash systems. We present a systematic framework for constructing MD-SC codes by optimally coupling individual SC codes together to attain fewer short cycles, and consequently better performance. For exchanging the connections, we follow three rules: (1) The connections involved in the highest number of certain cycles are targeted for rewiring; (2) The neighboring constituent 1D-SC code to which the targeted connections are rewired is chosen such that the minimum number of the certain cycles is attained; (3) The targeted connections are rewired to the same positions in the other constituent 1D-SC codes in order to preserve the low-latency decoding property.

The parity-check matrix $\mathbf{H}$ of a regular circulant-based (CB) code consists of $\kappa\gamma$ circulants, where γ is the column weight of the parity-check matrix and κ is the row weight. Each circulant is a $z \times z$ identity matrix cyclically shifted some units to the left. We use CB codes as the underlying block codes to construct SC codes. The parity-check matrix $\mathbf{H}_{\text{SC}}$ of a CB SC code is constructed as follows: First, the circulants of $\mathbf{H}$ are partitioned into $(m+1)$ disjoint component matrices (of the same size as $\mathbf{H}$): $\mathbf{H}_0, \mathbf{H}_1, \dots, \mathbf{H}_m$. Second, L copies of the component matrices are pieced together in a diagonal structure to obtain $\mathbf{H}_{\text{SC}}$ (see [7]). The parameters m and L are called the memory and the coupling length, respectively. A replica $\mathbf{R}_d$, $1 \leq d \leq L$, is a group of consecutive columns in $\mathbf{H}_{\text{SC}}$ that include one instance of each component matrix. We use the OO-CPO technique for designing SC codes [7]. In this work, we present a systematic framework to construct MD-SC codes, which is based on an optimal relocation of circulants to obtain notably lower population of short cycles (short cycles have a negative impact on the performance of graph-based codes).

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II. Novel Framework for MD-SC Code Design

Consider three instances of an SC code with parity-check matrix $\mathbf{H}_{\text{SC}}$, memory m, and coupling length L. Consider the middle replica $\mathbf{R}_d$ in $\mathbf{H}_{\text{SC}}$, where $d = \lceil L/2 \rceil$. Out of the non-zero circulants of this replica, we choose $\mathcal{T}$ circulants that are the most problematic, i.e., that contribute to the highest number of cycles-k, where k is the girth of the SC code. We relocate the chosen circulants to two auxiliary matrices, $\mathbf{P}$ and $\mathbf{Q}$, such that a relocated circulant from $\mathbf{H}_{\text{SC}}$ is moved to the same position in either $\mathbf{P}$ or $\mathbf{Q}$. The same relocations are repeated for all the $(L-1)$ remaining replicas. These auxiliary matrices have the same dimensions as $\mathbf{H}_{\text{SC}}$, and

$$\mathbf{H}_{\text{SC}} = \mathbf{H}'_{\text{SC}} + \mathbf{P} + \mathbf{Q}, \tag{1}$$

where $\mathbf{H}'_{\text{SC}}$ is derived from $\mathbf{H}_{\text{SC}}$ by removing the $\mathcal{T}$ chosen circulants. The parity-check matrix of the MD-SC code, $\mathbf{H}^{\text{MD}}_{\text{SC}}$, is then constructed as follows:

$$\mathbf{H}^{\text{MD}}_{\text{SC}} = \begin{bmatrix} \mathbf{H}'_{\text{SC}} & \mathbf{Q} & \mathbf{P} \\ \mathbf{P} & \mathbf{H}'_{\text{SC}} & \mathbf{Q} \\ \mathbf{Q} & \mathbf{P} & \mathbf{H}'_{\text{SC}} \end{bmatrix}. \tag{2}$$

Fig. 1 shows the graphical illustration of an MD-SC code having the presented structure. In our new MD-SC code design framework, we effectively answer two questions: which circulants to relocate, and where to relocate them. To this end, we first investigate the effect of relocating a subset of circulants of a cycle, and we call this subset targeted circulants. As we show, some relocations remove a cycle from $\mathbf{H}^{\text{MD}}_{\text{SC}}$, while others preserve the cycle.

Theorem 1. *Let* $C_{\mathcal{O}_k} = \{\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \dots, \mathcal{C}_{i_k}\}$ *be the sequence of circulants in* $\mathbf{H}_{\text{SC}}$ *that are visited in a clockwise order by*

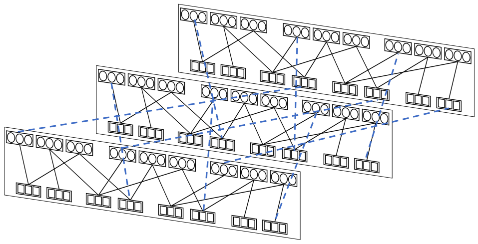


Fig. 1: An MD-SC code constructed by rewiring $\mathcal{T} = 1$ group of connections per replica. Constituent SC codes have parameters $\kappa = 3$, $\gamma = 2$, $z = 3$, $m = 1$, and $L = 3$. Circles (resp., squares) represent VNs (resp., CNs). Each line represents a group of connections. Rewired connections are shown in dashed blue lines.

a cycle-k (a cycle with length k), and let $M : \{\mathcal{C}_i\} \rightarrow \{0, 1, 2\}$ be a mapping from a non-zero circulant in $\mathbf{H}_{\text{SC}}$ to an integer in $\{0, 1, 2\}$ such that the values 0, 1, and 2 correspond to "no relocation", "relocation to $\mathbf{P}$", and "relocation to $\mathbf{Q}$", respectively. If the following equation holds, three instances of the cycle, in the three constituent SC codes, are preserved in the MD-SC code,

$$\sum_{u=1}^{k} (-1)^u M(\mathcal{C}_{i_u}) = 0 \qquad \text{(mode 3)}. \tag{3}$$

Otherwise, three instances of the cycle, in three constituent SC codes, form a cycle of length $3k$ in the MD-SC code.

Proof. Proof is given in [1]. □

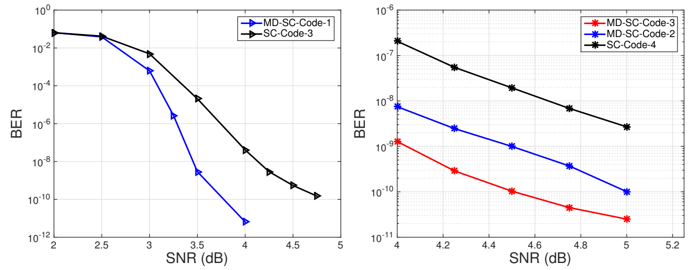
Next, we briefly explain our new systematic framework for constructing MD-SC codes which is based on a majority voting policy and aims at minimizing the population of cycles-k. Initially, $\mathbf{H}'_{\text{SC}} = \mathbf{H}_{\text{SC}}$ and $\mathbf{P} = \mathbf{Q} = \mathbf{0}$. At each iteration, one non-zero circulant in replica $\mathbf{R}_d$ of $\mathbf{H}'_{\text{SC}}$, $d = \lceil L/2 \rceil$, which is the circulant involved in the highest number of cycles-k in $\mathbf{H}^{\text{MD}}_{\text{SC}}$, is targeted for relocation. Each cycle $\mathcal{O}_k$ in $\mathbf{H}_{\text{SC}}$ that has the targeted circulant in its sequence votes for a subset of actions {relocate to $\mathbf{P}$, relocate to $\mathbf{Q}$, keep in $\mathbf{H}'_{\text{SC}}$} that remove instances of $\mathcal{O}_k$ in the MD-SC code, i.e., actions that break the condition in (3). The decision is made based on the majority of the votes. The same decision applies to the other $(L - 1)$ instances of this circulant in $\mathbf{H}'_{\text{SC}}$. The relocations are performed sequentially until the MD threshold is achieved, or relocation does not help anymore.

As the final (optional) step, a post-processing circulant power optimizer (PP CPO) is performed to remove as many as possible of remaining cycles-k in $\mathbf{H}^{\text{MD}}_{\text{SC}}$ by adjusting the powers of the relocated circulants.

III. Simulation Results

This work presented the first systematic framework for constructing MD-SC codes with better cycle properties, and AWGN channel is considered as the first step in the simulation results. One promising research direction is to investigate MD-SC codes on non-uniform channels, such as multilevel Flash.

Our code parameters are as follows: SC-Code-1 and SC-Code-2 are 1D-SC codes that are used as constituent SC codes. SC-Code-1 has column weight 4, length 2,890 bits, rate 0.74, and girth 6. SC-Code-2 has column weight 3, length 4,370 bits, rate 0.81, and girth 8. MD-SC-Code-1 and MD-SC-Code-2 are MD-SC codes with SC-Code-1 and SC-Code-2 as their constituent SC codes, and their MD densities are $\mathcal{T}_1 = 15$ (i.e., 22.06%) and $\mathcal{T}_2 = 12$ (i.e., 21.05%), respectively. MD-SC-Code-3 is constructed from MD-SC-Code-2 by applying the optional PP CPO step. MD-SC-Code-1 has length 8,670 bits and rate 0.74. MD-SC-Code-2 and MD-SC-Code-3 have length 13,110 bits and rate 0.81. SC-Code-3 (resp., SC-Code-4) is similar to SC-Code-1 (resp., SC-Code-2), but with larger coupling length, and it has comparable length and rate with MD-SC-Code-1 (resp., MD-SC-Code-2).



(a) (b)

Fig. 2: The BER performance comparison for: (a) MD-SC-Code-1 and SC-Code-3 (b) MD-SC-Code-2, MD-SC-Code-3, and SC-Code-4.

TABLE I: Comparison of the population of cycles of interest for MD-SC codes with their 1D-SC counterparts.

	SC-Code-3	MD-SC-Code-1
cycles-6	91,494	14,331

	SC-Code-4	MD-SC-Code-2	MD-SC-Code-3
cycles-8	1,034,609	280,968	253,851

According to TABLE I, MD-SC-Code-1 has nearly 84% fewer cycles-6 compared to SC-Code-3. Moreover, MD-SC-Code-2 and MD-SC-Code-3 have nearly 73% and 75% fewer cycles-8 compared to SC-Code-4, respectively. As Fig. 2(a) shows, MD-SC-Code-1 has nearly 3.8 orders of magnitude BER performance improvement at SNR= 4.0 dB and 1 dB SNR gain at BER= 10^{-10} compared to its 1D-SC counterpart, i.e., SC-Code-3. As Fig. 2(b) shows, MD coupling (resp., MD coupling along with PP CPO) results in nearly 1.3 (resp., 2.2) orders of magnitude BER performance improvement for MD-SC-Code-2 (resp., MD-SC-Code-3) compared to its 1D-SC counterpart, i.e., SC-Code-4, at SNR= 4.75 dB.

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