Syndrome-Coupled Rate-Compatible Error-Correcting Codes for Flash Memories

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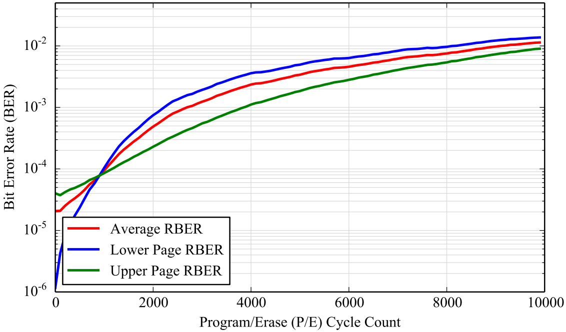
Outline

Motivation

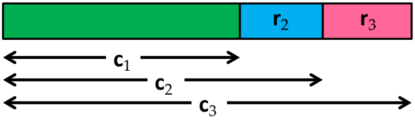
- Flash memory channels degrade with use.



- Stronger error correction capability is needed as wear increases.
- We propose a general code construction to address this need: **syndrome-coupled rate-compatible error-correcting codes**

Introduction to Rate-Compatible Codes

- Rate-compatible codes: a family of extended codes $\{\mathcal{C}_i\}_{i=1,\dots,M}$.
- Code $\mathcal{C}_{i+1}$ obtained from code $\mathcal{C}_i$ by adding redundancy symbols.
- Decreasing rate, but increasing error-correcting capability.
- Example: a family of 3-level rate-compatible codes.
  - Codes: $\mathcal{C}_1 \prec \mathcal{C}_2 \prec \mathcal{C}_3$.
  - Rates: $R_1 > R_2 > R_3$.



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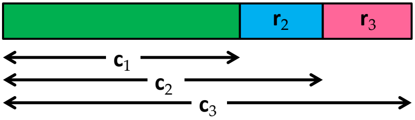

- Challenge: How to systematically add these redundancy symbols.

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- $\mathcal{C}_1^i = [n_1, n_1 - \sum_{m=1}^{i} v_m, d_i]_q$ for $1 \leq i \leq 3$, $d_1 \leq d_2 \leq d_3$.
- Parity-check matrices of $\mathcal{C}_1^1$, $\mathcal{C}_1^2$, and $\mathcal{C}_1^3$.

$$H_{\mathcal{C}_1^1} \qquad H_{\mathcal{C}_1^2} = \begin{bmatrix} H_{\mathcal{C}_1^1} \\ H_{\mathcal{C}_1^2|\mathcal{C}_1^1} \end{bmatrix} \qquad H_{\mathcal{C}_1^3} = \begin{bmatrix} H_{\mathcal{C}_1^1} \\ H_{\mathcal{C}_1^2|\mathcal{C}_1^1} \\ H_{\mathcal{C}_1^3|\mathcal{C}_1^2} \end{bmatrix}$$

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2) Choose two auxiliary nested codes $\mathcal{A}_2^3 \subset \mathcal{A}_2^2$.

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$$\boldsymbol{s}_2 = \boldsymbol{c}_1 H^T_{\mathcal{C}_1^2|\mathcal{C}_1^1} \quad \text{(compute syndrome } \boldsymbol{s}_2 \text{ of } \boldsymbol{c}_1\text{)}$$

$$\boldsymbol{a}_2^2 = \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2) \quad \text{(encode syndrome } \boldsymbol{s}_2\text{)}$$

$$\boldsymbol{c}_2 = (\boldsymbol{c}_1, \boldsymbol{a}_2^2) \quad \text{(append to } \boldsymbol{c}_1 \text{ to get } \boldsymbol{c}_2 \in \mathcal{C}_2\text{)}$$

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$\boldsymbol{\Lambda}_2^3 = \boldsymbol{a}_2^2 H^T_{\mathcal{A}_2^3|\mathcal{A}_2^2}$ (compute syndrome $\boldsymbol{\Lambda}_2^3$ of $\boldsymbol{a}_2^2$)

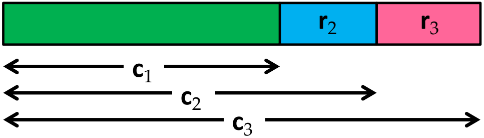
$\boldsymbol{a}_3^3 = \mathcal{E}_{\mathcal{A}_3^3}(\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3)$ (encode coupled syndrome $\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3$)

$\boldsymbol{c}_3 = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \boldsymbol{a}_3^3)$ (append to $\boldsymbol{c}_2$ to get $\boldsymbol{c}_3 \in \mathcal{C}_3$)

Encoding: Summary

- **Syndrome-coupled encoding**:

$$\begin{aligned} \boldsymbol{c}_1 &= \mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}) \\ \boldsymbol{c}_2 &= (\boldsymbol{c}_1, \boldsymbol{a}_2^2) = \left(\mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}), \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2)\right) \\ \boldsymbol{c}_3 &= (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \boldsymbol{a}_3^3) = \left(\mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}), \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2), \mathcal{E}_{\mathcal{A}_3^3}(\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3)\right) \end{aligned}$$



- Key idea: generate and encode syndromes progressively.
- Generalizes to any number of levels.
- Applicable to many codes, e.g., BCH, RS, LDPC, and Polar.

Minimum Distance

Theorem

Assume the auxiliary codes in the construction have minimum distances:

$$d_{min}(\mathcal{A}_2^2) = d_2 - d_1 \qquad d_{min}(\mathcal{A}_2^3) = d_3 - d_1$$
$$d_{min}(\mathcal{A}_3^3) = d_3 - d_2$$

Then, the syndrome-coupled codes $\mathcal{C}_i$, $1 \leq i \leq 3$, have length $N_i = \sum_{j=1}^{i} n_j$, dimension $k = n_1 - v_1$, and minimum distance d_i.

$\mathcal{C}_1^1 = [n_1, n_1 - v_1, d_1]$

$\mathcal{C}_1^2 = [n_1, n_1 - \sum_{i=1}^{2} v_i, d_2]$ $\quad \mathcal{A}_2^2 = [n_2, v_2, d_2 - d_1]$

$\mathcal{C}_1^3 = [n_1, n_1 - \sum_{i=1}^{3} v_i, d_3]$ $\quad \mathcal{A}_2^3 = [n_2, v_2 - \lambda_2^3, d_3 - d_1]$ $\quad \mathcal{A}_3^3 = [n_3, v_3 + \lambda_2^3, d_3 - d_2]$

- $\mathcal{C}_1 = [n_1, k, d_1]_q \prec \mathcal{C}_2 = [n_1 + n_2, k, d_2]_q \prec \mathcal{C}_3 = [n_1 + n_2 + n_3, k, d_3]_q$.

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Correctable Error-Erasure Patterns

- $\mathcal{C}_1 = [n_1, k, d_1]_q \prec \mathcal{C}_2 = [n_1 + n_2, k, d_2]_q \prec \mathcal{C}_3 = [n_1 + n_2 + n_3, k, d_3]_q$.

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- Transmit: $\boldsymbol{c}_3 = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \boldsymbol{a}_3^3) = \left(\mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}), \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2), \mathcal{E}_{\mathcal{A}_3^3}(\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3)\right) \in \mathcal{C}_3$.
- Receive: $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$, $\boldsymbol{y} \in (\mathbb{F}_q \cup \{?\})^{n_1+n_2+n_3}$.

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- For $1 \le i \le 3$, let t_i and τ_i denote the number of errors and erasures in the sub-block $\boldsymbol{y}_i$ of the received word $\boldsymbol{y}$.

Theorem

The code $\mathcal{C}_3$ can correct any combined error and erasure pattern that satisfies the following conditions:

$$\begin{aligned} 2t_1 + \tau_1 &\le d_3 - 1, \\ 2t_2 + \tau_2 &\le d_3 - d_1 - 1, \\ 2t_3 + \tau_3 &\le d_3 - d_2 - 1. \end{aligned}$$

Sequential Decoding

Transmit: $\boldsymbol{c}_3 = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \boldsymbol{a}_3^3) = \left(\mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}), \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2), \mathcal{E}_{\mathcal{A}_3^3}(\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3)\right)$.

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Output: $\hat{\boldsymbol{u}}$.

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- Examples: Polar codes on DMC; BCH, Reed-Muller codes on BEC.

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- Component codes (*capacity-achieving* with *specified rates*)

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- Decoding error probability over W_3
  Transmit: $\boldsymbol{c}_3 = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \boldsymbol{a}_3^3) = \left(\mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u}), \mathcal{E}_{\mathcal{A}_2^2}(\boldsymbol{s}_2), \mathcal{E}_{\mathcal{A}_3^3}(\boldsymbol{s}_3, \boldsymbol{\Lambda}_2^3)\right)$.
  Receive: $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$.
  Decoding: $\boldsymbol{y}_3 \to \boldsymbol{y}_2 \to \boldsymbol{y}_1$.

$$P_e^{(N_3)}(\mathcal{C}_3) \leq 1 - \left(1 - P_e^{(n_1)}(\mathcal{C}_1^3)\right)\left(1 - P_e^{(n_2)}(\mathcal{A}_2^3)\right)\left(1 - P_e^{(n_3)}(\mathcal{A}_3^3)\right) \to 0$$

Outline

Example: 2-Level BCH-Based Rate-Compatible Codes

- BCH component codes

  $\mathcal{C}_1^2 = [8191, 7398]_2$ $(t = 61)$ $\subset \mathcal{C}_1^1 = [8191, 7697]_2$ $(t = 38)$
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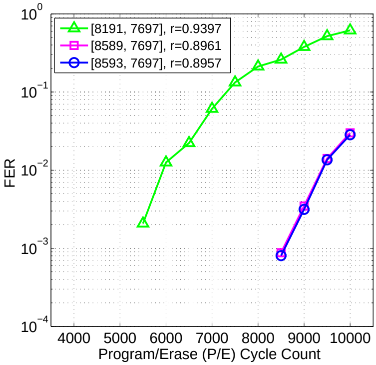
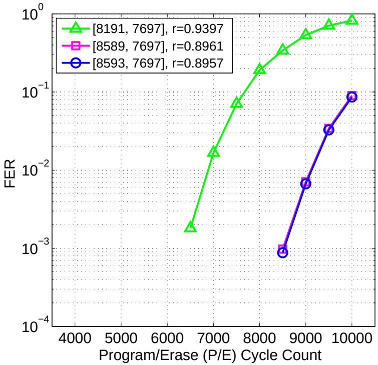
- For comparison, we use a shortened BCH code $\mathcal{C}_3 = [8593, 7697]_2$ $(t = 64)$, whose code length and rate are similar to those of $\mathcal{C}_2$.





- Left: lower page. The code $\mathcal{C}_2$ extends lifetime around 3500 cycles.
- Right: upper page. The code $\mathcal{C}_2$ extends lifetime around 2000 cycles.
- The code $\mathcal{C}_2$ is comparable to the shortened BCH code $\mathcal{C}_3$.

Example: 2-Level LDPC-Based Rate-Compatible Codes

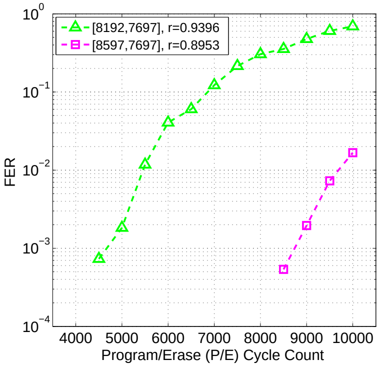
- LDPC component codes
  - Use Reed-Solomon based construction[1].
  - Nested and 4-cycle free structure.
  - $\mathcal{C}_1^2 \subset \mathcal{C}_1^1$ are two nested binary LDPC codes.
  - $\mathcal{C}_1^1$ is a $(4, 64)$-regular [8192,7697] LDPC code, $r = 0.9396$.
  - $\mathcal{C}_1^2$ is a $(7, 64)$-regular [8192,7400] LDPC code, $r = 0.9033$.
  - The auxiliary code $\mathcal{A}_2^2$ is a $(4, 15)$-regular [405,300] LDPC code.

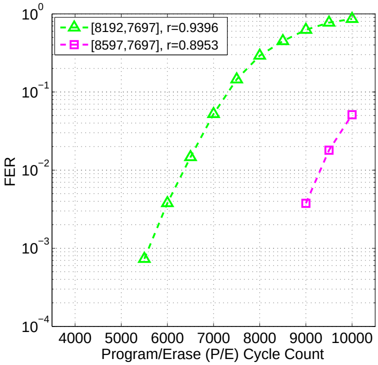
- 2-level rate-compatible codes

$$\mathcal{C}_1 = [8192, k = 7697]_2 \prec \mathcal{C}_2 = [8597, k = 7697]_2$$
$$R_1 = 0.9396 > R_2 = 0.8953$$

[1] Ryan and Lin, *Channel Codes: Classical and Modern*, Cambridge Univ. Press, 2009.

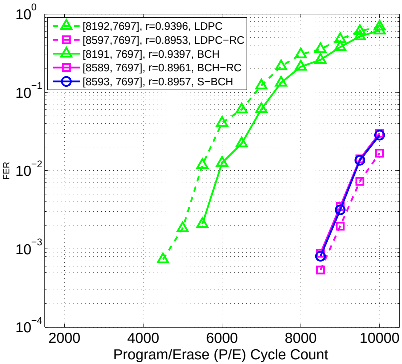
2-Level LDPC-Based Codes for MLC Flash Memories

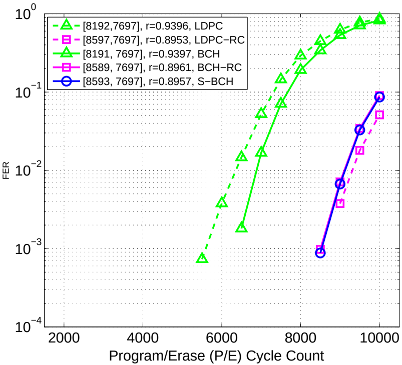




- Left: lower page. The code $\mathcal{C}_2$ extends lifetime around 4000 cycles.
- Right: upper page. The code $\mathcal{C}_2$ extends lifetime around 3000 cycles.

2-Level Rate-Compatible Codes for MLC Flash Memories





- At a higher rate of 0.94, the BCH code outperforms the LDPC code. The opposite conclusion holds for a lower rate of 0.90.

Outline

Summary

- Introduced rate-compatible error-correcting codes.
- Developed a new, general algebraic construction based upon syndrome-coupled encoding, with a sequential decoding algorithm.
- Studied the minimum distance properties and correctable error patterns of the codes.
- Proved the capacity-achieving property of our construction and equivalence to recent capacity-achieving polar code schemes.
- Applied 2-level rate-compatible codes to MLC flash memories.

Thanks