Syndrome-Coupled Rate-Compatible Error-Correcting Codes for Flash Memories

Pengfei Huang*, **Yi Liu***, **Xiaojie Zhang**†, **Paul H. Siegel***, and **Erich F. Haratsch** ‡
*Electrical and Computer Engineering Dept., University of California, San Diego, La Jolla, CA 92093 U.S.A
†CNEX Labs, San Jose, CA 95128 U.S.A ‡Seagate Technology, Fremont, CA 94538 U.S.A
{*pehuang,yil333,psiegel*}@*ucsd.edu*

I. Introduction

Rate-compatible error-correcting codes (ECCs), which consist of a set of extended codes, are of practical interest in both wireless communications and data storage. The idea of rate-compatible codes dates back to Davida and Reddy [1]. The most commonly used way to construct such codes is by puncturing [2], [4]. The second approach is by extending; recently, it was used to construct capacity-achieving rate-compatible polar codes [3].

In this work, we propose a general framework for constructing rate-compatible ECCs based on cosets and syndromes of a set of nested linear codes. We evaluate our construction from two points of view. From a combinatorial perspective, we show that we can construct rate-compatible codes with increasing minimum distances. From a probabilistic point of view, we also prove that we are able to construct capacity-achieving rate-compatible codes. Performance of two-level rate-compatible codes is evaluated for a multi-level cell (MLC) flash memory.

II. Definitions and Preliminaries

We use the notation $[n]$ to denote the set $\{1,\dots,n\}$. For a length-n vector $\boldsymbol{v}$ over $\mathbb{F}_q$ and a set $\mathcal{I} \subseteq [n]$, the operation $\pi_{\mathcal{I}}(\boldsymbol{v})$ denotes the restriction of the vector $\boldsymbol{v}$ to coordinates in the set $\mathcal{I}$. The transpose of a matrix H is written as H^T. A linear code $\mathcal{C}$ over $\mathbb{F}_q$ of length n, dimension k, and minimum distance d will be denoted by $\mathcal{C} = [n,k,d]_q$ or by $[n,k,d]_q$ for simplicity; in some cases, we will use notation $[n,k]_q$ to indicate only length and dimension.

Now, we present the definition of rate-compatible codes.

Definition 1. *For $1 \leqslant i \leqslant M$, let $\mathcal{C}_i$ be an $[n_i,k,d_i]_q$ linear code, where $n_1 < n_2 < \cdots < n_M$. The encoder of $\mathcal{C}_i$ is denoted by $\mathcal{E}_{\mathcal{C}_i} : \mathbb{F}_q^k \to \mathcal{C}_i$. These M linear codes are said to be M-level rate-compatible if for each i, $1 \leqslant i \leqslant M-1$, the following condition is satisfied for every possible input $\boldsymbol{u} \in \mathbb{F}_q^k$,*

$$\mathcal{E}_{\mathcal{C}_i}(\boldsymbol{u}) = \pi_{[n_i]}\Big(\mathcal{E}_{\mathcal{C}_{i+1}}(\boldsymbol{u})\Big). \tag{1}$$

We denote this M-level rate-compatible relation among these codes by $\mathcal{C}_1 \prec \mathcal{C}_2 \prec \cdots \prec \mathcal{C}_M$.

III. A General Construction for M-Level Rate-Compatible Codes

In our construction for M-level rate-compatible codes, we need a set of component codes which are defined as follows.

1) Choose a set of nested codes $\mathcal{C}_1^M \subset \mathcal{C}_1^{M-1} \subset \cdots \subset \mathcal{C}_1^1 = \mathcal{C}_1 = [n_1,k,d_1]_q$, where $\mathcal{C}_1^i = [n_1, n_1 - \sum_{m=1}^{i} v_m, d_i]_q$ for $1 \leqslant i \leqslant M$. We have $k = n_1 - v_1$ and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_M$. Define $\mathcal{C}_1^0 = \emptyset$ and for $1 \leqslant \ell \leqslant i$, let matrix $H_{\mathcal{C}_1^\ell|\mathcal{C}_1^{\ell-1}}$ represent a $v_\ell \times n_1$ matrix over $\mathbb{F}_q$ such that $\mathcal{C}_1^i$ has the following parity-check matrix:

$$H_{\mathcal{C}_1^i} = \begin{bmatrix} H_{\mathcal{C}_1^1} \\ H_{\mathcal{C}_1^2|\mathcal{C}_1^1} \\ \vdots \\ H_{\mathcal{C}_1^i|\mathcal{C}_1^{i-1}} \end{bmatrix}.$$

The encoder of code $\mathcal{C}_1$ is denoted by $\mathcal{E}_{\mathcal{C}_1} : \mathbb{F}_q^k \to \mathcal{C}_1$. We also use $\mathcal{E}_{\mathcal{C}_1}^{-1}$ as the inverse of the encoding mapping.

2) For the ith level, $2 \leqslant i \leqslant M$, consider a set of auxiliary nested codes $\mathcal{A}_i^M \subset \mathcal{A}_i^{M-1} \subset \cdots \subset \mathcal{A}_i^{i+1} \subset \mathcal{A}_i^i$, where $\mathcal{A}_i^j = [n_i, v_i + \sum_{m=2}^{i-1} \lambda_m^i - \sum_{\ell=i+1}^{j} \lambda_i^\ell, \delta_i^j]_q$ for $i \leqslant j \leqslant M$. Let matrix $H_{\mathcal{A}_i^i}$ represent an $(n_i - v_i - \sum_{m=2}^{i-1} \lambda_m^i) \times n_i$ matrix over $\mathbb{F}_q$ and matrix $H_{\mathcal{A}_i^\ell|\mathcal{A}_i^{\ell-1}}$, $i+1 \leqslant \ell \leqslant j$, represent a $\lambda_i^\ell \times n_i$ matrix over $\mathbb{F}_q$, such that $\mathcal{A}_i^j$ has the following parity-check matrix:

$$H_{\mathcal{A}_i^j} = \begin{bmatrix} H_{\mathcal{A}_i^i} \\ H_{\mathcal{A}_i^{i+1}|\mathcal{A}_i^i} \\ \vdots \\ H_{\mathcal{A}_i^j|\mathcal{A}_i^{j-1}} \end{bmatrix}.$$

For each $2 \leqslant i \leqslant M$, the encoder of code $\mathcal{A}_i^i$ is denoted by $\mathcal{E}_{\mathcal{A}_i^i} : \mathbb{F}_q^{v_i + \sum_{m=2}^{i-1} \lambda_m^i} \to \mathcal{A}_i^i$. We also use $\mathcal{E}_{\mathcal{A}_i^i}^{-1}$ as the inverse of the encoding mapping.

Note that we also define $\mathcal{C}_1^{M+1} = \emptyset$ and $\mathcal{A}_i^{M+1} = \emptyset$ for $2 \leqslant i \leqslant M$.

A. Construction and Minimum Distance

Now, we give a general algebraic construction for rate-compatible codes $\mathcal{C}_1 \prec \mathcal{C}_2 \prec \cdots \prec \mathcal{C}_M$ by using the nested component codes introduced above.

Construction 1: Encoding Procedure

Input: A length-k vector $\boldsymbol{u}$ of information symbols over $\mathbb{F}_q$.
Output: A codeword $\boldsymbol{c}_i \in \mathcal{C}_i$ over $\mathbb{F}_q$, for $i = 1,\dots,M$.

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c_1 = E_{C_1}(u).
s_i = c_1 H^T_{C_1^i | C_1^{i-1}} for i = 2,3,...,M.
for i = 2,...,M do
    a_i^i = E_{A_i^i}((s_i, Λ_2^i, ..., Λ_{i-1}^i)).
    c_i = (c_1, a_2^2, ..., a_i^i).
    for j = i+1,...,M do
        Λ_i^j = a_i^i H^T_{A_i^j | A_i^{j-1}}.
    end for
end for
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Where: $\boldsymbol{c}_1 = \mathcal{E}_{\mathcal{C}_1}(\boldsymbol{u})$; $\boldsymbol{s}_i = \boldsymbol{c}_1 H^T_{\mathcal{C}_1^i|\mathcal{C}_1^{i-1}}$; $\boldsymbol{a}_i^i = \mathcal{E}_{\mathcal{A}_i^i}\Big((\boldsymbol{s}_i, \Lambda_2^i, \cdots, \Lambda_{i-1}^i)\Big)$; $\boldsymbol{c}_i = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \cdots, \boldsymbol{a}_i^i)$; $\Lambda_i^j = \boldsymbol{a}_i^i H^T_{\mathcal{A}_i^j|\mathcal{A}_i^{j-1}}$.

We have the following theorem on the code parameters of the constructed rate-compatible codes $\mathcal{C}_1 \prec \mathcal{C}_2 \prec \cdots \prec \mathcal{C}_M$.

Theorem 2. *From Construction 1, the code $\mathcal{C}_i$, $1 \leqslant i \leqslant M$, has length $N_i = \sum_{j=1}^{i} n_j$ and dimension $K_i = k$. Moreover, assume that $\mathcal{A}_i^j$, $2 \leqslant i \leqslant M$ and $i \leqslant j \leqslant M$, has minimum distance $\delta_i^j \geqslant d_j - d_{i-1}$. Then the code $\mathcal{C}_i$ has minimum distance $D_i = d_i$.*

B. Decoding Algorithm and Correctable Error-Erasure Patterns

Assume a codeword $\boldsymbol{c}_M \in \mathcal{C}_M$, $\boldsymbol{c}_M = (\boldsymbol{c}_1, \boldsymbol{a}_2^2, \cdots, \boldsymbol{a}_M^M)$, is transmitted. Let the corresponding received word be $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_M)$ with errors and erasures, i.e., $\boldsymbol{y} \in (\mathbb{F}_q \cup \{?\})^{N_M}$, where the symbol ? represents an erasure. For $1 \leqslant i \leqslant M$, let t_i and τ_i denote the number of errors and erasures in the sub-block $\boldsymbol{y}_i$ of the received word $\boldsymbol{y}$.

The code $\mathcal{C}_M$ can correct any combined error and erasure pattern that satisfies the following condition:

$$\begin{aligned} 2t_1 + \tau_1 &\leqslant d_M - 1, \\ 2t_i + \tau_i &\leqslant \delta_i^M - 1, \ \forall\ 2 \leqslant i \leqslant M. \end{aligned} \tag{2}$$

To see this, we present a decoding algorithm, referred to as Algorithm 1, for $\mathcal{C}_M$. It uses the following component error-erasure decoders:

a) The error-erasure decoder $\mathcal{D}_{\mathcal{C}_1^i}$ for a coset of the code $\mathcal{C}_1^i$, for $1 \leqslant i \leqslant M$, is defined by

$$\mathcal{D}_{\mathcal{C}_1^i} : (\mathbb{F}_q \cup \{?\})^{n_1} \times (\mathbb{F}_q \cup \{?\})^{\sum_{j=1}^{i} v_j} \to \mathcal{C}_1^i + \boldsymbol{e} \cup \{\text{“e”}\}.$$

The decoder $\mathcal{D}_{\mathcal{C}_1^i}$ either produces a codeword in the coset $\mathcal{C}_1^i + \boldsymbol{e}$ or a decoding failure “e”.

b) The error-erasure decoder $\mathcal{D}_{\mathcal{A}_i^j}$ for a coset of the code $\mathcal{A}_i^j$, for $2 \leqslant i \leqslant M$ and $i \leqslant j \leqslant M$, is defined by

$$\mathcal{D}_{\mathcal{A}_i^j} : (\mathbb{F}_q \cup \{?\})^{n_i} \times (\mathbb{F}_q \cup \{?\})^{n_i - v_i - \sum_{m=2}^{i-1} \lambda_m^i + \sum_{\ell=i+1}^{j} \lambda_i^\ell} \to \mathcal{A}_i^j + \boldsymbol{e} \cup \{\text{“e”}\}.$$

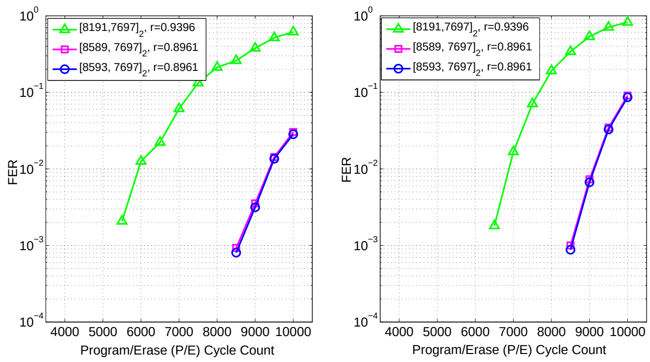
The decoder $\mathcal{D}_{\mathcal{A}_i^j}$ either produces a codeword in the coset $\mathcal{A}_i^j + \boldsymbol{e}$ or a decoding failure “e”.

Now, we present the decoding algorithm as follows.

Algorithm 1: Decoding Procedure for $\mathcal{C}_M$

Input: received word $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_M)$.
Output: A length-k vector $\boldsymbol{u}$ of information symbols over $\mathbb{F}_q$ or a decoding failure “e”.

for $i = M, M-1, \ldots, 2$ **do**
  Let the syndrome $\Lambda_i^i = \boldsymbol{0}$.
  $\hat{\boldsymbol{a}}_i^i = \mathcal{D}_{\mathcal{A}_i^M}\left(\boldsymbol{y}_i, (\Lambda_i^i, \Lambda_i^{i+1}, \cdots, \Lambda_i^M)\right)$.
  $(\boldsymbol{s}_i, \Lambda_2^i, \cdots, \Lambda_{i-1}^i) = \mathcal{E}_{\mathcal{A}_i^i}^{-1}(\hat{\boldsymbol{a}}_i^i)$.
end for
Let the syndrome $\boldsymbol{s}_1 = \boldsymbol{0}$.
$\boldsymbol{c}_1 = \mathcal{D}_{\mathcal{C}_1^M}\big(\boldsymbol{y}_1, (\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_M)\big)$.
Output $\boldsymbol{u} = \mathcal{E}_{\mathcal{C}_1}^{-1}(\boldsymbol{c}_1)$ if all above steps are successful; otherwise, return “e”.



(a) (b)

Fig. 1. FER performance of two-level rate-compatible codes for an MLC flash memory: (a) lower page and (b) upper page.

Theorem 3. *The code $\mathcal{C}_M$ can correct any combined error and erasure pattern that satisfies the condition in (2) by using Algorithm 1.*

Remark 1. Construction 1 can also be used to construct capacity-achieving rate-compatible codes by choosing proper component codes.

IV. Two-Level Rate-Compatible Codes for MLC Flash Memories

We design two-level rate-compatible codes based on binary BCH codes. We choose two nested binary BCH codes $\mathcal{C}_1^2 = [8191, 7398]_2 \subset \mathcal{C}_1^1 = [8191, 7697]_2$ as our component codes; the codes $\mathcal{C}_1^1$ and $\mathcal{C}_1^2$ can correct 38 and 61 errors, respectively. We also choose an auxiliary shortened BCH code $\mathcal{A}_2^2 = [398, 299]_2$, which can correct 11 errors. Then, from Construction 1, we obtain two-level rate-compatible codes $\mathcal{C}_1 = [8191, 7697]_2 \prec \mathcal{C}_2 = [8589, 7697]_2$. We apply $\mathcal{C}_1$ and $\mathcal{C}_2$ to an MLC flash memory and evaluate their performance.

The frame error rate (FER) performance of the constructed codes $\mathcal{C}_1$ and $\mathcal{C}_2$ for the lower page and upper page of an MLC flash memory are shown in Fig. 1(a) and Fig. 1(b), respectively. Compared to $\mathcal{C}_1$, the code $\mathcal{C}_2$ extends the lifetime around 3500 program/erase (P/E) cycles for the lower page and around 2000 P/E cycles for the upper page.

In addition, we evaluate a shortened BCH code $\mathcal{C}_{SBCH} = [8593, 7697]_2$, whose code length and rate are similar to those of the code $\mathcal{C}_2$. The code $\mathcal{C}_{SBCH}$ can correct 64 errors, and its FER performance for the lower page and upper page are shown in Fig. 1(a) and Fig. 1(b), respectively. It is shown that the FER of $\mathcal{C}_2$ is comparable to that of $\mathcal{C}_{SBCH}$.

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